Interest and annuities

Version 1.1

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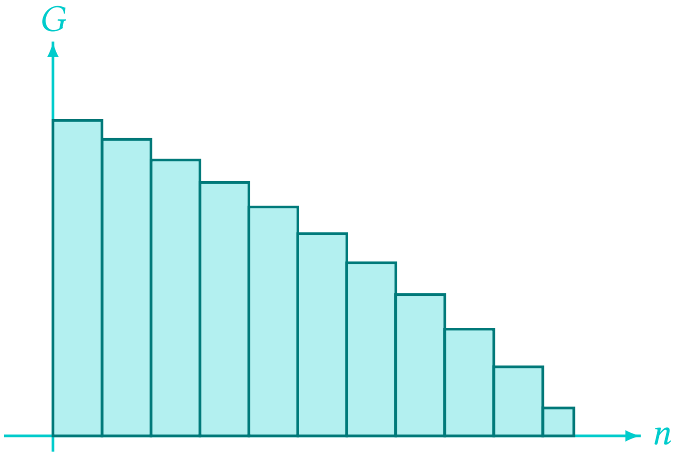

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Version 1.1, 2019

These notes are a translation of the Danish "Renter og annuiteter" written for the Danish stx.

They cover the basic theory on compound interest, savings and loans.

The document is written using the document preparation system LaTeX, see www.tug.org and www.miktex.org. Figures and diagrams are produced using *pgf/TikZ*, see www.ctan.org/pkg/pgf.

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1 Percent and interest

1.1 Preliminary concepts

The word *percent* comes from the latin *centum*, which means “hundred”, so percent means “per hundred”. So e.g. 3% means “3 per hundred”, i.e. 3 hundredths or, to put it another way,

$$3\% = \frac{3}{100} = 0.03\ .$$

In calculations, we can therefore always replace the symbol % by a division by 100.[1]

[1]We should always do this because it removes a lot of confusion concerning the meaning of % in the calculation.

If we want to write a certain number as a percentage, we perform the opposite calculation. E.g.

$$0.72 = 0.72 \cdot \frac{100}{100} = \frac{0.72 \cdot 100}{100} = \frac{72}{100} = 72\%\ .$$

In this calculation, we multiply the number 0.72 by $\frac{100}{100}$, which is equal to 1.[2] The calculation looks a bit elaborate; if we note that $\frac{100}{100}$ is actually 100%, we can instead write

[2]A number does not change when it is multiplied by 1—not even if the 1 is written as $\frac{100}{100}$.

$$0.72 = 0.72 \cdot 100\% = 72\%\ .$$

Here, we multiply 0.72 by 100%, which is just another way of multiplying by 1.

We often use the idea of percent to talk about fractions of a given quantity. The following theorem shows how to calculate percentages:

Theorem 1.1

$p\%$ of a given quantity K_0 is

$$p\% \cdot K_0 = \frac{p}{100} \cdot K_0\ .$$

Example 1.2 How much do you save if the price of a £80 jacket is lowered by 30%?

You will save 30% of £80, i.e.

$$30\% \cdot £80 = \frac{30}{100} \cdot £80 = 0.3 \cdot £80 = £24.$$

We may also want to know, how large a quantity is in relation to another. This is done in the following way:

Theorem 1.3

To find out, how many percent the quantity K_1 is in relation to the quantity K_0, we calculate

$$\frac{K_1}{K_0} \cdot 100\% \ .$$

Example 1.4 How large a percentage is 23 people out of 362?

To find out, we calculate

$$\frac{23}{362} = 0.0635 = 0.0635 \cdot 100\% = 6.35\% \ .$$

Example 1.5 How many percent is 465 out of 276?

Answer:

$$\frac{465}{276} = 1.6848 = 1.6848 \cdot 100\% = 168.48\% \ .$$

Here, we get a result above 100%, but this makes perfect sense since 465 is more than 276, so it has to be more than 100%.[3]

[3]It is important to remember that it is not the size of the numbers that determines, which number get divided by which; we always divide by the number with which we compare.

1.2 Growth in percent

In the preceding section, we looked at how to find a percentage of a given quantity, and how to compare two quantities. Here, we look at growth. How do we calculate the result if some quantity grows by a certain percent? And how do we find out, how much larger (or smaller) some quantity is in relation to another?

Calculated example If we want to add 12% to 140, we can do it in the following way:

1. First, we find 12% of 140:

$$12\% \cdot 140 = 0.12 \cdot 140 = 16.8 \ .$$

2. Then we add this to the original 140 and get:

$$140 + 16.8 = 156.8 \ .$$

This is actually a very elaborate way of doing it, especially if we want to add a certain percentage severeal times—e.g. if we want to find out, how much money is in a bank account after 1, 2, 3 or more years.

It turns out that the calculation above can be simplified quite a bit. If we combine the two steps, we see that in order to add 12% to 140, we need to calculate:

$$140 + 12\% \cdot 140 = 140 + 0.12 \cdot 140 \ .$$

If we factor out 140, we get

$$140 + 0.12 \cdot 140 = 140 \cdot (1 + 0.12) \ .$$

Here we se that in order to add 12% to 140, we need to multiply 140 by $1 + 0.12 = 1.12$.

We therefore have the following definition:

Definition 1.6

1. The *growth rate* r is the fraction (with sign) which a certain quantity grows.
2. The *multiplication factor* a is defined by

$$a = 1 + r.$$

Here are two examples of how to calculate the growth rate and the multiplication factor.

Example 1.7 If some quantity grows by 7.5% the growth rate is

$$r = 7.5\% = 0.075 \ ,$$

and the multiplication factor is

$$a = 1 + r = 1 + 0.075 = 1.075 \ .$$

Example 1.8 If some quantity *decreases* by 11%, the growth rate is

$$r = -11\% = -0,11 \ ,$$

and the multiplication factor is

$$a = 1 + (-0,11) = 0,89 \ .$$

Here, it is important to notice that when a quantity decreases, the growth rate is negative.

If we want to add 12% to 140 like we did before, we calculate

$$140 + 12\% \cdot 140 = 140 \cdot 1.12 \ .$$

This means that we actually just multiply 140 with the multiplication factor—which in this case is $a = 1.12$.

From this we get

Theorem 1.9

If we want to add a percentage to a quantity K_0, we get the new value

$$K_1 = K_0 \cdot a \ ,$$

where $a = 1 + r$ is the multiplication factor.

In calculations concerning growth in percent—or when we want to know how much bigger/smaller a certain quantity is in relation to another—we never use the growth rate r directly, but instead use the multiplication factor a.

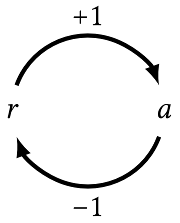

Figure 1.1: Conversion between growth rate and multiplication factor.

In any given context, we usually know only the growth rate r. We therefore need to calculate the multiplication factor a before proceeding with the calculations. If we instead are looking for a growth rate, we calculate the multiplication factor, and then convert this to a growth rate (see figure 1.1).

Here is an example where we add a percentage to a given quantity, using the theorem above:

Example 1.10 We want to add 25% VAT to an item, which costs DKK 399.96. What is the final price of this item?

Here, the growth rate is $r = 25\% = 0.25$. The multiplication factor is then

$$a = 1 + r = 1 + 0.25 = 1.25 \ .$$

The final price is therefore

$$399.96 \cdot 1.25 = 499.95 \ .$$

So the item costs DKK 499.95.

Now we look at how to compare two quantities:

Example 1.11 The price of an item drops from DKK 179.95 to 139.95. How many percent has the price decreased?

We can rewrite the formula $A_1 = A_0 \cdot a$ as $a = \frac{A_1}{A_0}$. Using this we calculate

$$a = \frac{A_1}{A_0} = \frac{139.95}{179.95} = 0.7777 \ .$$

This is the multiplication factor, which means that the growth rate is

$$r = a - 1 = 0.7777 - 1 = -0.2223 = -22.23\% \ .$$

The price has therefore dropped by 22.23%.[4]

[4] In this example, we see clearly that we calculate using the multiplication factor, but present the growth rate in the conclusion.

1.3 Compound interest

In the preceeding section, we explained how to add a certain percentage to a given quantity. Sometimes we need to add a percentage several times. E.g. if we deposit money into a bank account, how much money do we have in 2, 3 or more years?

Calculated example If we deposit DKK 10 000 into a bank account at an interest rate of 2.5% p.a.,[5] how much money do we have after 5 years?

[5] *per annum*, i.e. per year

According to theorem 1.9, we need to multiply the DKK 10 000 by the multiplication factor $a = 1 + 2.5\% = 1.025$ to find the amount of money after 1 year. This amount we multiply by 1.025 to find the amount after 2 years, etc. All in all, we need to multiply 10 000 by 1.025 five times:

$$10\,000 \cdot 1.025 \cdot 1.025 \cdot 1.025 \cdot 1.025 \cdot 1.025\ .$$

This means that we need to calculate

$$10\,000 \cdot 1.025^5 = 11\,314.08\ ,$$

so after 5 years, we have DKK 11 314.08 in our account.

Adding interest in this way is called compounding. The formula for *compound interest* is given in the following theorem.

Theorem 1.12: Compound interest

A quantity K_0 (called the *principal*), which increases by a growth rate of r per *compounding period*, increases to K_n after n periods, where

$$K_n = K_0 \cdot a^n\ ,$$

and $a = 1 + r$ is the multiplication factor. This formula can also be written as

$$K_n = K_0 \cdot (1 + r)^n\ .$$

We can answer several types of questions using this formula. A few are shown in the following examples.

Example 1.13 If we deposit \$ 10 000 into a bank account, what does the interest rate need to be p.a. for this amount to increase to \$ 12 000 in 10 years?

Here we know the principal $K_0 = 10\,000$, the amount $K_{10} = 12\,000$, and the number of periods (years) $n = 10$. If we insert these values into the formula for compound interest, we get

$$K_n = K_0 \cdot a^n \qquad \Rightarrow \qquad 12\,000 = 10\,000 \cdot a^{10}\ .$$

Now we solve the equation, and get

$$12\,000 = 10\,000 \cdot a^{10} \quad \Leftrightarrow \quad \frac{12\,000}{10\,000} = a^{10} \quad \Leftrightarrow \quad \sqrt[10]{\frac{12\,000}{10\,000}} = a.$$

a is then

$$a = \sqrt[10]{\frac{12\,000}{10\,000}} = 1.0184\ .$$

This is a multiplication factor, and we need the corresponding growth rate

$$r = 1.0184 - 1 = 0.0184 = 1.84\%\ .$$

If we want \$ 10 000 to increase to \$ 12 000 in 10 years, the interest rate needs to be 1.84% p.a.

Example 1.14 In a certain city, the growth rate has been 3% per year for the last 20 years. If there are 27 541 inhabitants in the city now, how many were there 12 years ago?

To solve this problem, we use the formula for compound interest and set $n = -12$, since we are going *back* 12 years.

We have $K_0 = 27\,541$, $r = 3\%$, i.e. $a = 1.03$, and $n = -12$. The formula then gives us the answer:

$$K_{-12} = 27\,541 \cdot 1.03^{-12} = 19\,317 \ .$$

12 years ago, the city had 19 317 inhabitants.

Example 1.15 Denmark has 5.6 million inhabitants. The growth rate is currently about 0.4% per year.[3] If this rate stays constant, how many years will pass before there are 6 million Danes?

In millions, $K_0 = 5.6$. The multiplication factor is $a = 1 + 0.4\% = 1.004$, and $K_n = 6$. The number of years, n, is unknown. We insert the known quantities into the formula for compound interest and get

$$K_n = K_0 \cdot a^n \qquad \Longrightarrow \qquad 6 = 5.6 \cdot 1.004^n.$$

We solve the equation:[6]

[6]Remember that the equation $a^x = b$ has the solution $x = \frac{\log(b)}{\log(a)}$.

$$\begin{aligned} 6 &= 5.6 \cdot 1.004^n && \Longleftrightarrow \\ \frac{6}{5.6} &= 1.004^n && \Longleftrightarrow \\ \frac{\log\left(\frac{6}{5.6}\right)}{\log(1.004)} &= n && \Longleftrightarrow \\ 17.3 &= n \ . \end{aligned}$$

In 17.3 years, the population will be 6 million if the growth rate stays at 0.4%.

1.4 Changing compounding periods

In this section, we will see how to convert between multiplication factors (and, by extension, growth rates) for different compounding periods, e.g. years and months.

Calculated example If a given quantity increases by 0.5% monthly, how do find the corresponding percentage increase per year?

The formula for compound interest tells us that in order to add interest to K_0 for *one* month, we need to multiply by the multiplication factor

$$a_{\text{month}} = 1 + 0.5\% = 1.005 \ .$$

Now, if we go forward one year, this corresponds to 12 months. According to the formula, we then need to multiply K_0 by

$$a_{\text{month}}^{12} = 1.005^{12} \ .$$

The multiplication factor for one year is therefore

$$a_{\text{year}} = 1,005^{12} = 1,0617 \ .$$

This corresponds to the growth rate $r_{\text{year}} = 1.0617 - 1 = 6.17\%$. A monthly rate of 0.5% thus corresponds to an annual rate of 6.17%.

Generalising this calculation gives us the following theorem:

Theorem 1.16

If the annual multiplication factor is a_{year}, and the monthly multiplication factor is a_{month}, then

$$a_{\text{year}} = a_{\text{month}}^{12} \ .$$

This theorem may be extended to other conversions than the one between months and years. The following examples illustrate this point.

Example 1.17 If some quantity K_0 grows by 4% per year, how many percent does it grow in 10 years?

The annual multiplication factor is $a_{1 \text{ year}} = 1 + 4\% = 1.04$. We can then calculate the multiplication factor for 10 years:

$$a_{10 \text{ years}} = a_{1 \text{ year}}^{10} = 1.04^{10} = 1.4802 \ .$$

This corresponds to a growth rate of

$$r_{10 \text{ years}} = 1.4802 - 1 = 0.4802 = 48.02\% \ .$$

If a quantity grows by 4% per year, it will grow by 48.02% in 10 years.

It is also possible to use theorem 1.16 to convert from years to months—i.e. in the opposite direction:

Example 1.18 If a bank pays 2.1% interest p.a., then what is the monthly interest rate?

First, we notice that since there are 12 months in a year, 1 month must correspond to $\frac{1}{12}$ of a year. Then we use theorem 1.16 like this:

$$a_{\text{month}} = a_{\text{year}}^{\frac{1}{12}} \ .$$

If we insert the annual growth rate $a_{\text{year}} = 1 + 2.1\% = 1.021$, we get

$$a_{\text{month}} = 1.021^{\frac{1}{12}} = 1.00173 \ .$$

An annual interest rate of 2.1% therefore corresponds to a monthly interest rate of 0.173%.

1.5 Average growth rate

In every preceding section, we looked at a fixed growth rate. If the growth rate changes, what is it then possible to say about the growth?

Calculated example We deposit \$ 1000 into an account. The money stays in the account for 3 years, during which time the interest rate changes as shown in table 1.2.

To determine how much money is in the account after 3 years, we need to know the corresponding multiplication factors. These are shown in the last column of table 1.2.

Table 1.2: The interest r_i and the multiplication factor a_i for an account over 3 years.

Year	r_i	a_i
1	2.7%	1.027
2	3.0%	1.030
3	1.5 %	1.015

Now we can calculate the amount:

$$\begin{aligned} K_3 &= K_0 \cdot a_1 \cdot a_2 \cdot a_3 \\ &= 1000 \cdot 1.027 \cdot 1.030 \cdot 1.015 = 1073.68 \; . \end{aligned} \tag{1.1}$$

So after 3 years, we have \$ 1073.68 in our account.

If the growth rate was fixed, what would it have to be in order for the account to have the same balance after 3 years? This fixed growth rate is an *average growth rate.*

To answer this question, we need to look at the formula for compound interest. If we denote the *average multiplication factor* by $\overline{a}$, then

$$K_3 = K_0 \cdot \overline{a}^3 \; .$$

Since this calculation must have the same result—$K_3 = 1073.68$—as (1.1), we must have

$$\overline{a}^3 = a_1 \cdot a_2 \cdot a_3 \; .$$

From this we get

$$\overline{a} = \sqrt[3]{a_1 \cdot a_2 \cdot a_3} \; .$$

In this case, the average multiplication factor becomes

$$\overline{a} = \sqrt[3]{1.027 \cdot 1.030 \cdot 1.015} = 1.02398 \; .$$

The corresponding average growth rate is

$$\overline{r} = \overline{a} - 1 = 1.02398 - 1 = 2.398\% \; .$$

In general, we have the following theorem:

Theorem 1.19: Average growth rate

If a quantity increases through n periods by the different multiplication factors $a_1, a_2, \ldots, a_n$, the average multiplication factor per period is

$$\overline{a} = \sqrt[n]{a_1 \cdot a_2 \cdot \cdots \cdot a_n} \; .$$

The average growth rate is therefore

$$\overline{r} = \sqrt[n]{(1 + r_1) \cdot (1 + r_2) \cdot \cdots \cdot (1 + r_n)} - 1 \; .$$

Table 1.3: Growth rate r_i and multiplication factor a_i for the population in a town in year i.

i	r_i	a_i
1	2.2%	1.022
2	3.1%	1.031
3	−1.2%	0.988
4	4.2%	1.042
5	6.0%	1.060

Example 1.20 The population of a town grows by changing rates in a 5-year period. The growth rates and factors can be seen in table 1.3.

r_3 is negative. This means that the population decreases in that year.

To find the average growth rate during the 5 years, we calculate the 5 multiplication factors (see table 1.3).

Now we can calculate the average multiplication factor:

$$\overline{a} = \sqrt[5]{1.022 \cdot 1.031 \cdot 0.988 \cdot 1.042 \cdot 1.060} = 1.02832 \,.$$

The average growth rate during the 5 years is then

$$\overline{r} = 1.02832 - 1 = 0.02832 = 2.832\% \,.$$

1.6 Exercises

Exercise 1.1
Write these numbers as a decimal:

a) 3% b) 6.12% c) 36.7%
d) 89.2% e) 142% f) 3251%

Exercise 1.2
Write these numbers as a percentage:

a) 0.14 b) 0.587 c) 1.043
d) 4.12 e) 0.051 f) 0.0093

Exercise 1.3
How much is

a) 21% of 5613? b) 6.5% of 10 214?
c) 141 % of 45.32?

Exercise 1.4
How many percent is

a) 45 of 89? b) 34 of 27?
c) 12 of 6539? d) 0.32 of 21?

Exercise 1.5
A furniture store has a chair on sale for kr. 398. The normal price is kr. 585.

How many percent do you save?

Exercise 1.6
An empty soda can weighs 25 g. It is filled with 330 g of soda.

How many percent of the total weight is the weight of the can?

Exercise 1.7
An empty soda bottle weighs 253 g, and it is filled with 250 g of soda.

How many percent of the total weight is the bottle?

Exercise 1.8
In 1970, a preserve tin weighed 68.4 g, and in 1993 it only weighed 56.7 g.

How many percent did the tins weigh less in 1993 than in 1970?

Exercise 1.9
In 1950, a beer can weighed 90.7 g, whereas in 1993, beer cans only weighed 17 g.

a) How many percent did the cans weigh more in 1950 compared to 1993?

b) How many percent did the cans weigh less in 1993 than in 1950?

Exercise 1.10
The Earth orbits the Sun in an ellipse-shaped orbit. When the Earth is closest to the Sun (at the beginning of January), the distance is 147.1 million km. When the Earth is farthest from the Sun (at the beginning of July), the distance is 152.1 million km.

a) How many percent is the largest distance larger than the smallest?

b) How many percent is the smallest distance smaller then the largest?

Exercise 1.11
The Moon orbits the Earth in an ellipse-shaped orbit. When the Moon is closest to the Earth, the distance is 363 300 km. When the Moon is farthest away from the Earth, the distance is 405 500 km.

a) How many percent is the largest distance larger than the smallest?

b) How many percent is the smallest distance smaller then the largest?

Exercise 1.12
When you pay for goods or work, you pay VAT. The VAT is 25%. You receive a bill of kr. 6250 excluding VAT.

a) What is the total price you have to pay?

Another bill is for kr. 9456,90 including VAT.

b) What was the price before the VAT was added?

Exercise 1.13
A company offers a discount of 10%. When the company writes the bill, they can choose to subtract the discount before or after the VAT is added.

Does it matter for the customer in which order this happens?

Exercise 1.14
Kr. 10 000 is put into an account with an annual interest of 2.5%.

How much money is in the account after 5 years?

Exercise 1.15
After 7 years, an amount has increased to kr. 7994.18. The interest is 3% annually.

How much money was deposited into the account initially?

Exercise 1.16
Kr. 150 000 is deposited into an account. After 3 years this amount has increased to kr. 178 652.40.

What was the annual interest?

Exercise 1.17
Kr. 7000 is deposited into an account at an annual interest of 1.5%. After a number of years, the account holds kr. 7429.54.

How many years have passed?

Exercise 1.18
During a given period of time, salaries increased by 2% every 6 months. How many percent did the salaries increase in 3 years?

Exercise 1.19
A car salesman advertises loans with an interest of 2.5% per month.

How much does this correspond to in annual interest?

Exercise 1.20
A bank advertises a savings account with an annual interest of 3.6%.

How much is the interest per month?

Exercise 1.21
A company stock increased its value in the first three years by 13%, 23% and 34% per year. The following year, the value decreased by 17%.

How many percent did the value of the stock increase per year on average?

Exercise 1.22
The value of a rare coin increases by 78% in 11 years.

What is the average annual value increase in percent?

Exercise 1.23
During the years 2005–2010, a company's production increased by 5% per year. How many percent was the production larger in 2010 compared to 2005?

Exercise 1.24
Three years ago, the CEO of a certain company said that the goal of the company was to increase its value by 10% per year over a five-year period. The first three years, the value of the company only increased by 4.5% per year.

How much must the company increase its value on average for the next two years to reach this goal?

2 Index numbers

In the fields of economy and statistics, we often use the so-called *index numbers* to represent data. Index numbers reflect the value of a given quantity in relation to the value for a certain year, called the *base year*. The index number for a certain year is the ratio of the value for that year and the base value.

If a quantity has increased by 13% compared to the base year, the index number is 113—while a decrease of 5% compared to the base year will have index 95.

Table 2.1: Average annual income, 2012–2016.[4]

Year	Income (DKK)
2012	288 684
2013	293 979
2014	298 785
2015	308 144
2016	312 649

Calculated example The average annual income for Danes over the age of 14 (for 5 consecutive years) is listed in table 2.1.

We can describe these numbers as index numbers with base year 2012. The value in 2012 (DKK 288 684) is the index 100; and the remaining indices are calculated from the value in 2012.

The ratio between the average annual income in 2013 and in 2012 is

$$\frac{293\,979}{288\,684} = 1.018 = 101.8\% \,.$$

The index in 2012 is therefore 101.8.

The average annual income in 2014 compared to that in 2012 yields

$$\frac{298\,785}{288\,684} = 1.035 = 103.5\% \,,$$

i.e. the index is 103.5 in 2014. Notice that we do *not* write the % sign.

We calculate the remaining indices in the same way and get the numbers in table 2.2. If we choose another base year, we get other values for the indices. With 2014 as the base year, we get table 2.3.

Table 2.2: Index numbers for the average annual income with base year 2012.

Year	Index
2012	100
2013	101.8
2014	103.5
2015	106.7
2016	108.3

Table 2.3: Index numbers for the average annual income with base year 2014.

Year	Index
2012	96.6
2013	98.4
2014	100
2015	103.1
2016	104.6

2.1 Changes in percent

When we talk about changes in index numbers, it is important to distinguish between two things: Changes in percent, and changes in *percentage points*. We cover the last term first.

Percentage points The change in percentage points is the absolute difference of the two indices, we compare.

E.g., if we look at the indices with base year 2012 from the example above, we see that the difference of the indices from 2016 and 2014 is

$$108.3 - 103.5 = 4.8 .$$

The index has increased by 4.8 percentage points from 2014 to 2016.

The change in percentage points depends on which year we have chosen to be the base year.

Change in Percent If we want to find the relative change (i.e. the change in percent) form 2014 to 2016, we need to compare the two numbers by calculating

$$\frac{108.3}{103.5} = 1.046 .$$

This is a multiplication factor. The corresponding growth rate is $1.046 - 1 = 0.046$. This means that from 2014 til 2016 the average income increases by 4.6%. The calculations above show that is important to distinguish between growth in percent or percentage points.

The following two examples show how to perform som useful calculations using index numbers:

Table 2.4: Denmark's production of biogas energy from 2012 to 2016, with 2012 as base year.[6]

Year	Index
2012	100
2013	107.9
2014	128.9
2015	158.0
2016	225.6

Example 2.1 (Absolute numbers) In table 2.4, Denmark's energy production from biogas is listed for the years 2012 to 2016, with 2012 as the base year.

If we are told that the production of biogas energy was 1328 GWh in 2014, we can find the energy production for another year by multiplying the production in 2014 with the multiplication factor from that year to 2014.

The multiplication factor from 2012 to 2014 is $\frac{100}{128.9}$, so the production of biogas energy in 2012 was

$$\frac{100}{128.9} \cdot 1328 = 1030 ,$$

i.e. 1030 GWh.

Table 2.5: Denmark's production of biogas energy, 2012–2016.

Year	Biogas energy (GWh)
2012	1030
2013	1112
2014	1328
2015	1627
2016	2324

We calculate the production in 2013 in the same way:

$$\frac{107.9}{128.9} \cdot 1328 = 1112 .$$

So, the production of biogas energy was 1112 GWh in 2013.

Calculating the missing values for the remaining three years gives us table 2.5.

Example 2.2 (Shift of base) Here, we analyse the same numbers as in example 2.1, see table 2.4.

Sometimes we need to shift the base year. This could be the case if we want to compare two different indices that initially have different base year.

If we want to recalculate the table to have base year 2014, we again calculate the relative value compared to the base value. The base year is now 2014, so for 2012 we get

$$\frac{100}{128.9} = 0.776 .$$

I.e. the index for 2012 is now 77.6.

For 2013, we get

$$\frac{107.9}{128.7} = 0.837 ,$$

so the index is 87.3.

The new index numbers are listed in table 2.6.

Table 2.6: Denmark's production of biogas energy, with 2014 as the base year.

Year	Index
2012	77.6
2013	83.7
2014	100
2015	122.5
2016	175.0

2.2 Exercises

Exercise 2.1
The table lists index numbers for the price of a Kung Fu (ice cream) for three different years.[7]

Year	Index
1977	19.4
1999	100
2014	166.7

In 2014, a Kung Fu cost kr. 15.

a) What was the price of the ice cream in 1977 and 1999?

Exercise 2.2
The average consumer price of petrol in January of several years is listed in the table:[5]

Year	Price (kr.)
1995	5.80
2000	7.85
2005	8.55
2010	10.53
2015	10.79
2019	11.59

a) Calculate the index numbers for the consumer price of petrol from 1995 to 2019 with 2000 as the base year.

b) Determine the increase in percent from 2005 to 2015 and from 2010 to 2019.

Exercise 2.3
The table list the populations of Funen, Jutland and Zealand for some of the years from 1976 to 2019.[2]

Year	Funen	Jutland	Zealand
1976	412 450	1 894 541	2 163 685
1990	426 106	1 982 364	2 134 816
2000	439 608	2 067 637	2 235 839
2010	454 358	2 160 878	2 348 684
2019	469 724	2 258 499	2 530 093

a) Determine the index numbers of the populations of Funen, Jutland and Zealand with 2000 as the base year.

b) Calculate which of the populations has increased by the largest percentage from 1976 to 2019.

Exercise 2.4
The table below lists the average alcohol consumption per Dane, in litres and as index numbers with 2000 as the base year.[1]

Year	Consumption (litres)	Index
2000	9,5	100
2005		96,8
2010		88.4
2015	7.8	

a) Determine the alcohol consumption per Dane in 2005.
Determine the index number for 2015.

b) How many percent did the consumption of alcohol decrease on average from 2000 to 2015?

3 Annuities

If we regularly make payments to a savings account in a bank, we cannot use compound interest to calculate the amount of money in our account. This is because in a savings account, we make more than one payment—usually we make payments at fixed intervals.

If we have a loan, we also need other formulas, because here we make regular payments, but the loan also accumulates interest at regular intervals.

3.1 Savings annuity

A savings annuity consists of periodic payments into an account, which also accumulates interest.

If we have a fixed interest rate, it is possible to derive a formula that tells us, how much money is in the account after a certain number of payments.

Calculated example Here, we look at a savings account with an interest rate (growth rate) of 1.5% p.a. Each year we make a payment of £1000. If the payments start on 2 January 2018, and are made on 2 January each year, the date is 2 January 2021 after the first 4 payments. So, we have saved money for three years, but the amount of payments is 4. If we want to calculate the amount of money in the account, we may do it the following way:

The amount inserted each year is the *payment*, b. The annual growth rate is denoted by r, the number of payments is n, and the balance of the account is A_n. In the example given we have

$$b = 1000\ , \qquad r = 0.015 \qquad \text{and} \qquad n = 4\ .$$

The quantity A_4, which is the balance of our account after the 4 payments, is unknown—this is what we want to calculate. If we look at what happens to our payments, we have

2 January 2018 we make a payment of £1000. This amount accumulates interest until 2 January 2021, in which time it increases to $1000 \cdot 1.015^3$ (using the formula for compound interest).

2 January 2019 we again make a payment of £1000, which accumulates interes for 2 years until 2 January 2021, in which time it has increased to $1000 \cdot 1.015^2$.

2 January 2020 we make a payment of £1000, which accumulates interest for 1 year and increases to $1000 \cdot 1.015$.

2 January 2021 we make a payment of £1000.

We find the total balance of our account by adding the 4 values above, i.e.

$$A_4 = 1000 + 1000 \cdot 1.015 + 1000 \cdot 1.015^2 + 1000 \cdot 1.015^3 \ .$$

Here we can factor out 1000 to get

$$A_4 = 1000 \cdot (1 + 1.015 + 1.015^2 + 1.015^3) = 4090.90 \ . \tag{3.1}$$

So, this is the balance of our account after 4 payments. The £4000 are the payments, we have made, while the remaining £90.90 is accumulated interest.

In this way, we can calculate the balace of a savings account, but the calculations are very time consuming. E.g. if we want to calculate the balance after 15 payments the parenthesis in the calculation (3.1) is quite unwieldy. It turns out there is a formula:

Theorem 3.1: Savings annuity

For a savings annuity, we have

$$A_n = b \cdot \frac{a^n - 1}{r},$$

where A_n is the balace after the nth payment, b is the payment, r is the growth rate, and $a = 1 + r$ is the multiplication factor. We can also write this formula as

$$A_n = b \cdot \frac{(1 + r)^n - 1}{r}.$$

The proof of this formula is given in section 3.3 below.

Example 3.2 Each year £2000 is paid into a savings account with interest rate 1.3%. If we make 10 payments, then

$$b = 2000 \ , \qquad r = 0.013 \ , \qquad a = 1.013 \qquad \text{and} \qquad n = 10 \ .$$

Immediately after the 10th payment, the balance is

$$A_{10} = 2000 \cdot \frac{1.013^{10} - 1}{0.013} = 21\,211.50 \ .$$

I.e. the balance of the account is £21 211.50.

If the payment is unknown, it can be calculated like this:

Example 3.3 If we want to save £5000 in 15 annual payments, and the interest rate is 2.1%, how large do the payments need to be?

In this case, we know

$$r = 0.021 \ , \qquad a = 1.021 \ , \qquad n = 15 \qquad \text{and} \qquad A_{15} = 5000 \ .$$

We insert these into the formula and get

$$\begin{aligned} 5000 &= b \cdot \frac{1.021^{15} - 1}{0.021} & \Longleftrightarrow \\ 5000 \cdot \frac{0.021}{1.021^{15} - 1} &= b & \Longleftrightarrow \\ 287.05 &= b. \end{aligned}$$

So, if we want to save £5000 in 15 annual payments, each payment needs to be £287.05.

Hvis antallet af indbetalinger er ukendt, bliver udregningen en smule mere besværlig:

If the number of payments is unknown, the calculations look like this:

Example 3.4 If we have a savings account with a fixed interest rate of 2.3% and save up £600 each year, how many payments do we need to make, before our account holds £7500?

Here, we know

$$b = 600 \;, \qquad r = 0.023 \;, \qquad a = 1.023 \qquad \text{and} \qquad A_n = 7500 \;.$$

We insert this into the formula to get

$$7500 = 600 \cdot \frac{1.023^n - 1}{0.023} \;.$$

We solve this equation using a CAS, which gives us the solution

$$n = 11.11 \;.$$

So, it is not quite enough to make 11 payments, we need 12. We could of course have found the same result by trial and error with different values of n.

3.2 Annuity loans

Now we look at what happens when we take out a loan. A lot of loans are paid back in periodic payments, which cover both installment and interest. The initial debt (i.e. the money we borrow) is called the *principal.*

Exactly how such an *annuity loan* works is described in this example:

Calculated example 2 January 2018, we take out a loan of £2000. The interest rate is 9%, and the payments are £300 annually. The payments are made 2 January, and on this date interest is also added. We continue payments until the loan is paid back in full. Now, the following happens:

2 January 2019 9% interest is added, so the debt increases to

$$2000 \cdot 1.09 = 2180 \;.$$

Next, we subtract the payment of £300, which brings the remaining debt down to

$$2180 - 300 = 1880\,.$$

Note that eventhough we paid £300, our debt has only decreased by £120.

2 January 2020 Again 9% interest is added (to the remaining debt of £1880); our debt is now

$$1880 \cdot 1.09 = 2049.2\,.$$

From this amount we again subtract our payment of £300, and the remaining debt is

$$2049.2 - 300 = 1749.2\,.$$

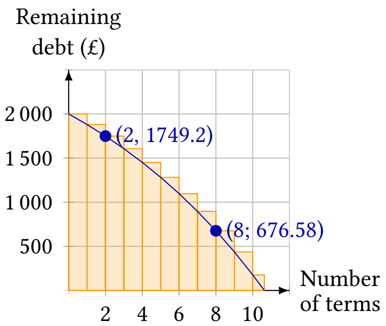

Figure 3.1: A loan of £2000 with 9% interest is paid back in annual payments of £300.

We continue to pay back our loan in this manner, until the remaining debt is 0. We find that the number of terms is approximately 11, so we actually pay back $11 \cdot 300 = 3300$—which means we pay back £3300 for a £2000 loan. Figure 3.1 shows how the remaining debt changes with time.

The following formulas apply to annuity loans:

Theorem 3.5: Annuity loan

For an annuity loan we have

$$G = y \cdot \frac{1 - a^{-n}}{r} \qquad \text{and} \qquad y = G \cdot \frac{r}{1 - a^{-n}},$$

where G is the principal, y is the payment, r is the growth rate, n is the number of terms, and $a = 1 + r$ is the multiplication factor.

We can also write these formulas as

$$G = y \cdot \frac{1 - (1 + r)^{-n}}{r} \qquad \text{and} \qquad y = G \cdot \frac{r}{1 - (1 + r)^{-n}}.$$

The proof of this theorem is given in section 3.3 below.

Example 3.6 (Unknown principal) If we can afford to make monthly payments of £50 for 3 years, and the annual interest rate is 8%, what size loan can we afford?

Here, the annual payments are $12 \cdot 50 = 600$. So,

$$y = 600\,, \qquad r = 0.08\,, \qquad a = 1.08 \qquad \text{and} \qquad n = 3\,.$$

Theorem 3.5 tells us that the principal is

$$G = 600 \cdot \frac{1 - 1.08^{-3}}{0.08} = 1546.26$$

Thus, we can afford a loan of £1546.26.

But remember that the amount we are actually paying back is $3 \cdot £600 = £1800$.

The situation in this example might seem a bit unreal. Usually people know exactly how much money they need to borrow. But theorem 3.5 can also be used to calculate the payments if we know the size of the loan and the number of terms needed to pay it back.

Example 3.7 (Unknown payment) If we borrow £5000 and want to pay it back over 5 years, what must the payments be at an interest rate is 12%?

Here, we know

$$G = 5000\ , \qquad r = 0.12\ , \qquad a = 1.12 \qquad \text{and} \qquad n = 5.$$

Inserting these into the formula from theorem 3.5 yields

$$y = 5000 \cdot \frac{0.12}{1 - 1.12^{-5}} = 1387.05\ .$$

This is the amount we need to pay back annually. The corresponding monthly payment is then

$$y_{\text{monthly}} = \frac{1387.05}{12} = 115.59\ ,$$

i.e. a monthly payment of £115.59.

3.3 Proof of the annuity formulas

In this section, we are going to prove the formulas in theorems 3.1 and 3.5.

Proof (of theorem 3.1)
If we look at the equation (3.1), we can generalise this to get

$$A_n = p \cdot (1 + a + a^2 + \cdots + a^{n-1}).$$

We now define

$$S = 1 + a + a^2 + \cdots + a^{n-1}.$$

So, $A_n = p \cdot S$. We also have

$$a \cdot S = a \cdot (1 + a + a^2 + \cdots + a^{n-1}) = a + a^2 + a^3 + \cdots + a^n.$$

If we now calculate $(a - 1) \cdot S$, we get

$$\begin{aligned}(a-1)\cdot S &= a \cdot S - S \\ &= (a + a^2 + a^3 + \cdots + a^n) - (1 + a + a^2 + \cdots + a^{n-1}) \\ &= a^n - 1\end{aligned}$$

This means that

$$(a-1)\cdot S = a^n - 1 \qquad \Longleftrightarrow \qquad S = \frac{a^n - 1}{a - 1},$$

and since $a - 1 = r$, we have

$$S = \frac{a^n - 1}{r}.$$

We already know that $A_n = p \cdot S$, therefore

$$A_n = p \cdot \frac{a^n - 1}{r},$$

which proves the theorem. ∎

The proof of theorem 3.5 uses theorem 3.1 and the formula for compound interest.

Proof (of theorem 3.5)
When we pay back a debt in n equal payments y, the value of these payments can be calculated using the formula for a savings annuity. If we call this quantity K, we have

$$K = y \cdot \frac{a^n - 1}{r}.$$

But according to the formula for compound interest, the value of the debt after n terms must be

$$K = G \cdot a^n.$$

These two expressions for K must be equal, i.e.

$$\begin{aligned} G \cdot a^n &= y \cdot \frac{a^n - 1}{r} &\Leftrightarrow \\ G &= y \cdot \frac{a^n - 1}{r \cdot a^n} &\Leftrightarrow \\ G &= y \cdot \frac{\frac{a^n}{a^n} - \frac{1}{a^n}}{r} &\Leftrightarrow \\ G &= y \cdot \frac{1 - a^{-n}}{r}. \end{aligned}$$

This proves the theorem. ∎

3.4 Exercises

Exercise 3.1
Let $A_n = 1200$ kr., $r = 2\%$ and $n = 8$.

Use the savings annuity formula to determine b.

Exercise 3.2
Let $b = 1273$ kr., $r = 11.11\%$ and $n = 7$.

Use the savings annuity formula to determine A_n.

Exercise 3.3
Let $A_n = 6170$ kr., $r = 4\%$ and $b = 193$ kr..

Use the savings annuity formula to determine n.

Exercise 3.4
Let $A_n = 3000$ kr., $b = 300$ kr. and $n = 8$.

Use the savings annuity formula to determine r to two decimals. (Write an equation and use a CAS to solve it.)

Exercise 3.5
In an annuity savings account, the monthly deposit is kr. 1620 and the monthly interest is 1%.

How much money is in the account after the 9th deposit?

Exercise 3.6
A family is able to deposit £800 into a savings account each month. The monthly interest is 0.75%.

For how many months must the family make these deposits for the account to exceed £25 000?

Exercise 3.7
A family wants to save for the 10% down payment of a house which costs £400 000. Their bank offers a monthly interest of 0.4%.

How much do they need to deposit each month if they want to be able to afford the down payment after 48 deposits?

Exercise 3.8
Gerald has made quarterly deposits of £405. After 27 quarterly deposits, his account has a balance of £15 000.

a) What has the quarterly interest been (2 decimals)?

b) What is the corresponding annual interest rate?

Exercise 3.9
A person make annual deposits of £1835 into a savings account at an interest rate of 2.3%.

a) What is the balance in the account after the 28th deposit?

b) How much total interest has he received?

The person now moves his money to a different bank which offers a savings account with an interest rate, which is twice the rate of the old account.

c) Answer the two questions above for the new interest rate.

d) Does doubling the interest cause the total savings or the total accumulated interest to double?

Exercise 3.10
Neil decides to deposit £300 each month into an account with an interest rate of 0.8% per month.

a) How much will the balance be after the 15th deposit?

Neil has unforeseen expenses, so he only makes 10 deposits. Then he leaves the money for 5 months at the given interest rate.

b) What is the balance of the account now?

c) How much accumulated interest does Neil lose by not making the last 5 deposits?

Exercise 3.11
Let $G = 150\,000$ kr., $r = 2.5\%$ and $n = 14$.

Use the annuity loan formula to determine y.

Exercise 3.12
Let $y = 12\,000$ kr., $r = 4.5\%$ and $n = 28$.

Use the annuity loan formula to determine G.

Exercise 3.13
Let $G = 12\,345$ kr., $y = 719.74$ kr. og $r = 4.44\%$.

Use the annuity loan formula to determine n.

Exercise 3.14
Let $G = 12\,000$ kr., $y = 1000$ kr. and $n = 15$.

Use the annuity loan formula to determine r (2 decimals). (Write an equation and use a CAS to solve it.)

Exercise 3.15
Valerie loans money to buy a scooter. The scooter costs £2300, but she gives a down payment of 20%. She then pays the rest in 48 equal instalments.

How much must she pay per month if the monthly interest rate on the loan is 2.3%?

Exercise 3.16
At "Honest Henry's Bikes and Scooters" they offer an annuity loan for a mountain bike which costs £1700. The loan is paid in monthly intalments, and the monthly interest rate is 2%. The loan is paid off in 8 years.

a) Determine the monthly payment.

b) How much accumulated interest are you going to pay for this loan?

Exercise 3.17
Archibald Johnson is offered an annuity loan for £4500 with annual payments an an annual interest rate of 5.6%. He has to pay back the loan in 7 years.

Determine the monthly payment.

Exercise 3.18
How much can Penelope loan at the bank when she can afford to pay £200 every 6 months, the interest rate is 5%, and the loan must be paid back after 6 years?

Exercise 3.19
Abigail is getting married and buys a $8300 wedding dress. She can pay for the dress in 60 monthly instalments at a monthly interest rate of 2%—or in 72 intalments at a rate of 1.75% per month.

a) What will Abigail pay per month in each of these scenarios?

b) What will she pay in total for the dress in each case?

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