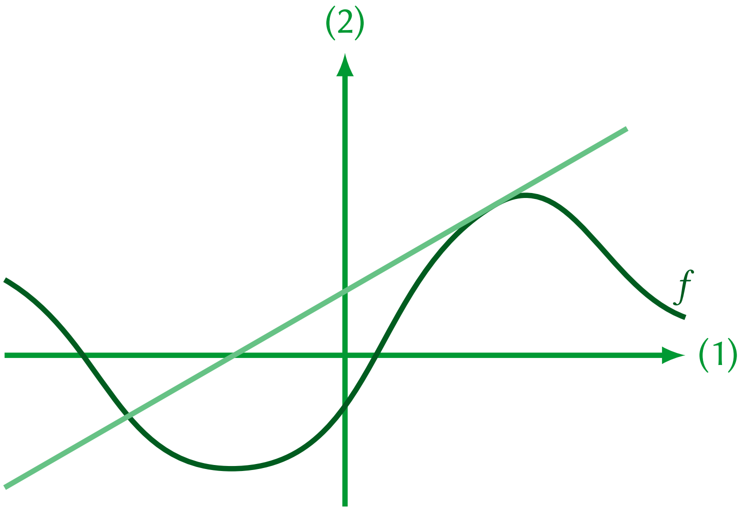
Differential Calculus

Version 1.0

November 18, 2019



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www.mathematicus.dk

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Version 1.0, 2019

These notes are a translation of the Danish "Differentialregning" written for the Danish stx.

The first chapter introduces the idea of limits and continuity. Because these topics are not an independent part of the curriculum, the chapter is kept very brief.

This document is written primarily for the Danish stx, but may be used freely for non-commercial purposes.

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Contents

1 Limits

When we investigate functions, it sometimes happens that the function is undefined for certain values of the independent variable.

Example 1.1 The function $g(x) = \frac{1}{x-3}$ is undefined for $x = 3$. The reason why the function is undefined here, is that when you try to calculate $g(3)$, you get

$$g(3) = \frac{1}{3-3} = \frac{1}{0} ,$$

which is undefined.

If we draw the graph of this function, we get figure 1.1. Here, it is clear that something special happens around $x = 3$, and that it does not make sense to talk about the function value $g(3)$.

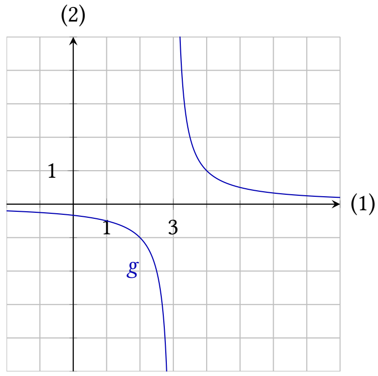


Figure 1.1: The graph of $g(x) = \frac{1}{x-3}$.

But functions exist which behave differently in places where they are not defined.

Example 1.2 Look at the function

$$f(x) = \frac{x^2 - 5x + 6}{x - 3} .$$

This function is undefined for $x = 3$ because

$$f(3) = \frac{3^2 - 5 \cdot 3 + 6}{3 - 3} = \frac{0}{0},$$

which has no meaning.

If we draw the graph of f, we get figure 1.2. Here, we see that even though the function is undefined for $x = 3$, the graph allows us to say what $f(3)$ should have been if this value were defined.

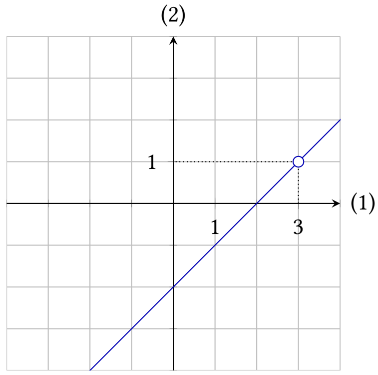


Figure 1.2: The graph of $f(x) = \frac{x^2-5x+6}{x-3}$.

If we input values of x which are "close to" 3, we get table 1.3. From figure 1.2 and table 1.3, it seems reasonable to claim that the closer x gets to 3, the closer $f(x)$ will get to 1.

Table 1.3: Function values for the function $f(x) = \frac{x^2-5x+6}{x-3}$.

x	$f(x)$
2.9	0.9
2.99	0.99
3	undefined
3.01	1.01
3.1	1.1

So, if we were to give $f(3)$ a value, it seems reasonable to choose 1—even though $f(3)$ actually is undefined.

Even if $f(3)$ is undefined for the function in example 1.2, $f(x)$ will approach a certain value as x approaches 3. We say that $f(x)$ has a *limit for x approaching 3*. This limit has the value 1. We write this as

$$\lim_{x \to 3} f(x) = 1 .$$

The limit of the function $f(x)$ as x approaches 3 is 1 because $f(x)$ gets closer and closer to 1 as x gets closer and closer to 3. If we are to give a precise mathematical definition of limits, we need to describe first what we mean by "close to". Therefore, we define a so-called *neighbourhood*:

Definition 1.3

Given a number a and a distance ε, we define the *open neighbourhood* $\Omega_\varepsilon(a)$ to be the open interval $]a - \varepsilon; a + \varepsilon[$.

The *punctured neighbourhood* $\Omega^\circ_\varepsilon(a)$ of a is the open neighbourhood $\Omega_\varepsilon(a)$ with the number a removed: $\Omega^\circ_\varepsilon(a) = \Omega_\varepsilon(a) \setminus \{a\}$.

The point is that the neighbourhood $\Omega_\varepsilon(a)$ contains all of the numbers closer to a than ε. If ε is a small number, the neighbourhood contains the numbers which are "close to" a—and if we let ε get smaller and smaller, we are looking at numbers which get closer and closer to a.

We need a *punctured* neighbourhood because when we look at a neighbourhood of x_0, the function we are looking at is not necessarily defined at this value of x. Therefore, we need a neighbourhood where this number is excluded.

We may then define the limit of a function at a given point in the following way:

Definition 1.4

Let a function f and a number L be given, and let $f(x)$ be defined for all x in a punctured neighbourhood of x_0.

If for any arbitrarily small open neighbourhood $\Omega_\varepsilon(L)$ of L, we can find a punctured neighbourhood $\Omega^\circ_\delta(x_0)$ of x_0 so that

$$x \in \Omega^\circ_\delta(x_0) \quad \Rightarrow \quad f(x) \in \Omega_\varepsilon(L) \,,$$

we call L the *limit of* $f(x)$ *for* x *approaching* x_0, and we write

$$\lim_{x \longrightarrow x_0} f(x) = L \,.$$

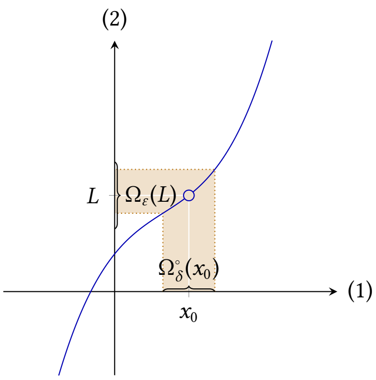


Figure 1.4: If the function value must be in the neighbourhood $\Omega_\varepsilon(L)$ of L, we can achieve this by letting x be in the small punctured neighbourhood $\Omega^\circ_\delta(x_0)$ of x_0.

Figure 1.4 shows how we may describe this: The function value $f(x)$ will be in the small neighbourhood $\Omega_\varepsilon(L)$ of L, as long as x is in the small punctured neighbourhood Ω°_δ of x_0. So, no matter how small a neighbourhood we choose around L, we can always find a small punctured neighbourhood around x_0, so that if x is in this small punctured neighbourhood, $f(x)$ will be in the chosen neighbourhood of L.

Since we can make the neighbourhood $\Omega_\varepsilon(L)$ arbitrarily small, this is the same as saying that when x gets closer and closer to x_0, $f(x)$ will get closer and closer to L.

Note that definition 1.4 does not consider a possible function value at $x = x_0$. I.e. the function might be defined for this value of x, or it might be undefined. If the function f is defined at x_0, a function value $f(x_0)$ exists, but the limit is defined completely independent of this value—and a possible function value does not even have to be equal to the limit.

If we want to investigate, how functions behave close to values of x where they *are* defined, what do we find then?

Example 1.5 What is $\lim_{x \to 5} x^2 + 3$?

The expression $x^2 + 3$ is defined for $x = 5$, where the expression yields

$$5^2 + 3 = 28 \ .$$

If we let x approach 5, the value of $x^2 + 3$ will approach 28, and we find

$$\lim_{x \to 5} x^2 + 3 = 28 \ .$$

Sometimes we can just input the x we want to investigate, into the function.

If we are looking for the limit of a function $f(x)$ as x approaches x_0, we can sometimes just calculate $f(x_0)$. However, this is not always the case. Sometimes it is not the case even when the function is defined for $x = x_0$.

Example 1.6 Here, we look at the function

$$f(x) = \begin{cases} x + 1 & \text{for } x < 2 \\ 4 - x & \text{for } x \geq 2 \end{cases} \ .$$

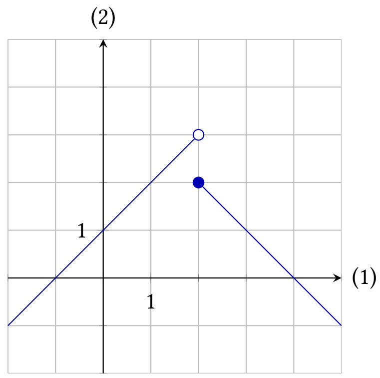


Figure 1.5: The graph of the piecewise linear function in example 1.6.

So, the function f behaves in such a way that it is equal to $x + 1$ as long as $x < 2$, and thereafter it is equal to $4 - x$. Therefore, we are looking at a piecewise linear function. The graph of f is shown in figure 1.5.

The function value at $x = 2$ is

$$f(2) = 4 - 2 = 2 \ ,$$

but what is the limit of this function as x approaches 2?

When $x < 2$, $f(x) = x + 1$, i.e. $f(x)$ will get closer and closer to $2 + 1$ when x gets closer to 2. If we investigate the limit by approaching $x = 2$ from below, we find the value 3.

But if we investigate the limit of $f(x)$ by approaching $x = 2$ from above, we follow the graph of $4 - x$, and then the function value approaches 2, when x approaches 2.

Because we get two different answers, depending on which way we approach $x = 2$, we conclude that the limit $\lim_{x \to 2} f(x)$ *does not exist*—even though the function itself is defined for $x = 2$.

Example 1.7 In example 1.2, we looked at the function $f(x) = \frac{x^2 - 5x + 6}{x - 3}$ and came to the conclusion that

$$\lim_{x \to 3} f(x) = 1 \ .$$

We could argue that this cannot be known just from looking at figure 1.2 and table 1.3, because it is impossbile to know whether the true limit is e.g. $1,00000326$ (and not exactly 1) based solely on the figure and the table.

However, we can rewrite $x^2 - 5x + 6$ as $(x-2)(x-3)$, and therefore

$$\frac{x^2 - 5x + 6}{x - 3} = \frac{(x-2)(x-3)}{x-3} = x - 2 \,,$$

as long as $x \neq 3$.

But because the definition of a limit is independent of how the function actually behaves when $x = 3$ and only depends on what happens when x is close to 3, we have

$$\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3} = \lim_{x \to 3} x - 2 \,.$$

Thus we have reduced the problem to figuring out which number $x - 2$ approaches when x approaches 3. This number is exactly 1.

So,

$$\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3} = 1 \,.$$

As the example above shows, it makes sense to investigate whether the function we are looking at, when we are looking for a limit, can be rewritten so that it is easier to see what the limit is.

Besides simplifying the expression, for which we are finding the limit, we can also use the following rules,[1] which we will not prove here:

Theorem 1.8

Let two functions f and g, and a constant k be given. If $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ exist, then

1. $\lim_{x \to x_0} k \cdot f(x) = k \cdot \lim_{x \to x_0} f(x)$
2. $\lim_{x \to x_0} f(x) + g(x) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$
3. $\lim_{x \to x_0} f(x) - g(x) = \lim_{x \to x_0} f(x) - \lim_{x \to x_0} g(x)$
4. $\lim_{x \to x_0} f(x) \cdot g(x) = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x)$.

If furthermore $\lim_{x \to x_0} g(x) \neq 0$, then

5. $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$.

1.1 Continuity

Most of the functions you have met so far, have graphs which are connected. Functions like that have such graphs are called *continuous*. So, a function is continuous (everywhere, or in an interval) if its graph has no "holes" (in the given interval). This is a somewhat loose definition, but continuity may be defined precisely using limits.

In example 1.6, we looked at a function whose graph *was not* connected (see figure 1.5). In the example, we showed that this function has no limit at the point where the graph "jumps".

However, in example 1.5 we looked at a function where the limit at a given x was exactly equal to the function value. The graph of *this* function is connected because when we approach the chosen x-value from above as well as from below, we will get the same function value—and this function value corresponds to a point on the graph.

We may therefore define continuous functions in the following way:

Definition 1.9

A function f is called *continuous* on an interval $]a;b[$ if for all $x_0 \in\ ]a;b[$

$$\lim_{x \longrightarrow x_0} f(x) = f(x_0) \ .$$

Example 1.10 The function

$$f(x) = x^2 + 4 \ ,$$

is continuous for all $x \in \mathbb{R}$.

If we choose e.g. $x_0 = 3$, we find

$$\lim_{x \longrightarrow 3} f(x) = \lim_{x \longrightarrow 3} x^2 + 4 = 3^2 + 4 = f(3) \ ,$$

and we may do this for any value of x_0—not just 3.

So, $f(x)$ is continuous.

Example 1.11 The function

$$f(x) = \begin{cases} x & \text{for } x \neq 3 \\ 4 & \text{for } x = 3 \end{cases}$$

is not continuous for all x.

We see from the graph (figure 1.6) that

$$\lim_{x \longrightarrow 3} f(x) = 3 \ ,$$

but

$$f(3) = 4 \ .$$

So, $\lim_{x \longrightarrow 3} f(x) \neq f(3)$, and the functions is not continuous for all x.

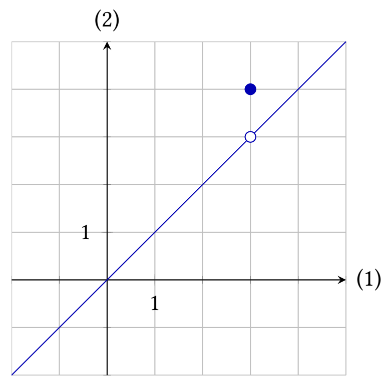


Figure 1.6: This function is not continuous.

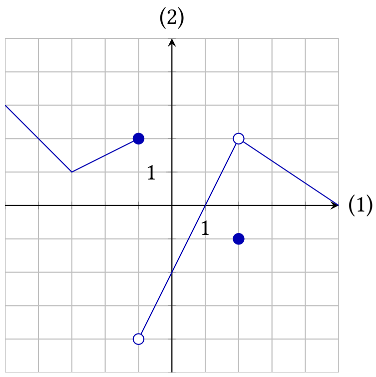
1.2 Exercises

Exercise 1.1
The function f has the formula

$$f(x) = \frac{x^2 - 1}{x - 1}\ .$$

a) Explain why $f(1)$ is undefined.
b) Calculate $f(0.9)$, $f(0.99)$ og $f(0.999)$.
c) Calculate $f(1.1)$, $f(1.01)$ og $f(1.001)$.
d) Estimate the value of $\lim_{x \to 1} f(x)$.

Exercise 1.2
The figure below shows the graph of the function f:



Investigate whether the following limits exist, and determine their values if they do:

a) $\lim_{x \to -3} f(x)$ b) $\lim_{x \to -1} f(x)$
c) $\lim_{x \to 1} f(x)$ d) $\lim_{x \to 2} f(x)$

Exercise 1.3
Draw the graphs of the following functions, and use the graph to determine whether the functions have a limit for $x \longrightarrow 1$:

a) $f(x) = \begin{cases} 3 - x & \text{for } x < 1 \\ 2x & \text{for } x \geq 1 \end{cases}$

b) $g(x) = \begin{cases} 2x - 1 & \text{for } x < 1 \\ 3x + 1 & \text{for } x \geq 1 \end{cases}$

c) $f(x) = \begin{cases} x + 3 & \text{for } x < 1 \\ 5 & \text{for } x = 1 \\ 2x + 2 & \text{for } x > 1 \end{cases}$

Exercise 1.4
Determine the following limits by calculation:

a) $\lim_{x \to 2} x^2 - 4$ b) $\lim_{x \to 0} \frac{x^2}{6x}$

c) $\lim_{x \to 5} \frac{x^2 - 10x + 25}{x - 5}$ d) $\lim_{x \to 2} \frac{x^2 + 5}{x^2 - 3}$

e) $\lim_{x \to -10} 8$ f) $\lim_{x \to 0} \frac{x - \sqrt{x}}{\sqrt{x}}$

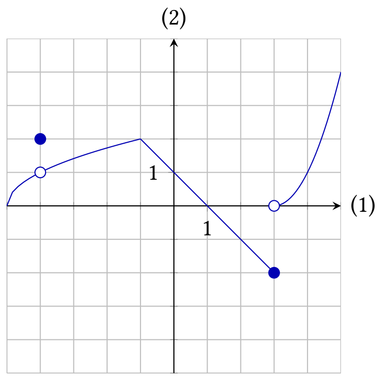
g) $\lim_{x \to 3} \frac{x - 3}{x^2 - 5x + 6}$

Exercise 1.5
Does the function

$$f(x) = \frac{x}{|x|}$$

have a limit for $x \longrightarrow 0$?

Exercise 1.6
The figure below shows the graph of the function f:



Is the function $f(x)$ continuous at

a) $x = -4$? b) $x = -2$?

c) $x = -1$? d) $x = 0$?

e) $x = 1$? f) $x = 3$?

Does the function have a limit at these x-values?

2 Derivatives

Differential calculus is a branch of mathematics concerned with describing how fast a function $f(x)$ increases for specific values of x. One way to investigate this is to look at how steep the graph is at the point $(x_0, f(x_0))$.

The usual measure of “steepness” is the *slope*; but since only straight lines have slopes, we need a way to express the behaviour of the graph at the point $(x_0, f(x_0))$ as a straight line, for which we can find the slope.

If the graph of the function is nice and smooth, we may draw a line at each point which lines up with the graph at this point. Such a line is called a *tangent*. An illustration of this is shown in figure 2.1.

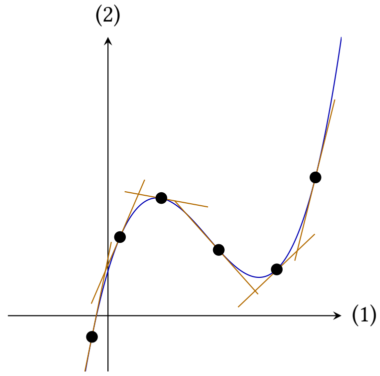


Figure 2.1: At each point on the graph we may draw a tangent. Here, some of the tangents are illustrated by line segments.

Example 2.1 Here, we look at the function $f(x) = 3x^2 + 7$. The graph of this function passes through the point $(5, 82)$. At this point, the graph has a tangent, see figure 2.2.

The slope of the tangent at this point is denoted by $f'(5)$. If we know in advance that $f'(5) = 30$, we can find an equation of the tangent.

The tangent is a straight line. So, the equation is $y = ax + b$. When we know that $f'(5) = 30$, we also know that the equation is $y = 30x + b$. The point of tangency is the point $P(5, 82)$, and therefore

$$82 = 30 \cdot 5 + b \qquad \Longleftrightarrow \qquad b = 82 - 30 \cdot 5 = -68 \ .$$

So, the tangent to the graph at the point $P(5, 82)$ has the equation

$$y = 30x - 68 \ .$$

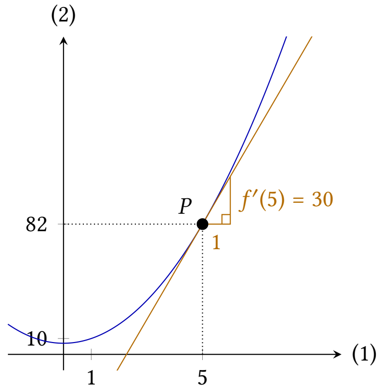


Figure 2.2: The graph f has a tangent at the point $(5, 82)$.

In the example above, we saw that it was possible to find an equation for a given tangent when we already knew the slope of the tangent. The question is now, how do we find this slope?

Of course, we might simply draw the graph and the tangent as best we can and then simply read the slope from the drawing; but this is hardly an exact method.

The tangents to the graph of a function are straight lines. To determine the slope of a straight line, we need two points on the line. Here, we have a problem: We only know one point, namely the point of tangency $P(x_0, f(x_0))$ which is the point at which the tangent touches the graph.

Because we do not know the equation of the tangent, we cannot find another point. The best we can do is to find another point on the graph $Q(x, f(x))$, which is close to the point of tangency P, see figure 2.3.

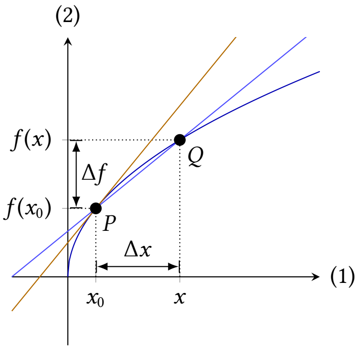


Figure 2.3: The graph of f passes through the two points P and Q. Q is not on the tangent but on the secant.

If we calculate the slope using the points $P(x_0, f(x_0))$ and $Q(x, f(x))$, we will not find the slope of the tangent, but the slope of a *secant*,[1] which is only an approximation. The closer x is to x_0, i.e. the closer Q is to P, the better the approximation will be, since the secant in figure 2.3 will move closer and closer to the tangent, the closer x is to x_0.

[1]A *secant* is a straight line passing through two points on the graph. It *intersects* the graph rather than just touching it as a tangent would.

So, we can find an approximation of the tangent slope $f'(x_0)$ by calculating the slope of the line through the two points P and Q, i.e.

$$f'(x_0) \approx \frac{f(x) - f(x_0)}{x - x_0},$$

where x is close to x_0.

But the aim is, of course, to find an exact value for the tangent slope and not just an approximation. We can do this by letting x approach x_0. We cannot just let $x = x_0$, because if we insert x_0 in place of x, we get

$$\frac{f(x_0) - f(x_0)}{x_0 - x_0} = \frac{0}{0},$$

which does not make sense. We therefore define $f'(x_0)$ to be the limit as $x \longrightarrow x_0$, i.e.

$$f'(x_0) = \lim_{x \longrightarrow x_0} \frac{\Delta f}{\Delta x}.$$

This limit is called the *derivative* of f at the point $(x_0, f(x_0))$.

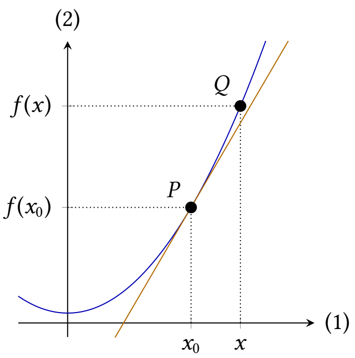


Figure 2.4: The point P is the point of tangency, so P lies on the tangent as well as on the graph. The point Q lies only on the graph.

Example 2.2 The graph of $f(x) = 3x^2 + 7$ passes through the point $P(x_0, f(x_0))$. The tangent to the graph of f at this point has slope $f'(x_0)$. To calculate this value, we first find the secant slope using the points P and Q, see figure 2.4.

The point Q has the coordinates $Q(x, f(x))$, so Δf is:[2]

$$\begin{aligned}\Delta f &= f(x) - f(x_0)\\ &= (3x^2 + 7) - (3x_0^2 + 7)\\ &= 3x^2 - 3x_0^2\\ &= 3(x + x_0)(x - x_0)\end{aligned}$$

[2]In the calculation, we use the identity

$$a^2 - b^2 = (a + b)(a - b).$$

Next, we calculate $\frac{f(x) - f(x_0)}{x - x_0}$:

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{3(x + x_0)(x - x_0)}{x - x_0} = 3(x + x_0).$$

The tangent slope (or the *derivative*) at the point is then the limit as $x \longrightarrow x_0$:

$$\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \longrightarrow x_0} 3(x + x_0) = 6x_0.$$

We therefore conclude that for the function $f(x) = 3x^2 + 7$,

$$f'(x_0) = 6x_0.$$

We sum up the method in the following definition:

Definition 2.3

For a function f we define the derivative $f'(x_0)$ to be

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} .$$

If this limit exists for every x in the open interval $]a; b[$, we say that f is *differentiable* in $]a; b[$.

Since $\Delta x = x - x_0$, $x = x_0 + \Delta x$, and therefore

$$\Delta f = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0) .$$

Definition 2.3 can therefore also be stated in this way:

Definition 2.4: Alternative definition

For a function f we define the derivative $f'(x_0)$ to be

$$f'(x_0) = \lim_{x \to x_0} \frac{\Delta f}{\Delta x} ,$$

where $\Delta f = f(x_0 + \Delta x) - f(x_0)$.

If this limit exists for every x in the open interval $]a; b[$, we say that f is *differentiable* in $]a; b[$.

When we look at the definition of the derivative, we see that the calculation of the derivative is done in three steps:

1. Calculate the function value increase Δf, and reduce this as much as possible.
2. Calculate the difference quotient $\frac{\Delta f}{\Delta x}$, and reduce as much as possible.
3. Determine the limit of $\frac{\Delta f}{\Delta x}$ as $x \to x_0$ (or $\Delta x \to 0$). This is the derivative $f'(x_0)$.

This is sometimes called the three-step method. However, it is important to note that because we are calculating a limit, the calculations are sometimes quite complex, and sometimes we even need to switch the steps around.

2.1 Differentiability

It turns out that whenever a function is differentiable, it is also continuous. So, a requirement for differentiability is that the graph has to be unbroken. The opposite is not true; it is possible to find functions whose graphs have no breaks, but which are not differentiable.

In plain terms, differentiability corresponds to the graph being a "smooth" curve. The graph cannot have "kinks". Figure 2.5 shows an example of a function which is continuous but not differentiable.

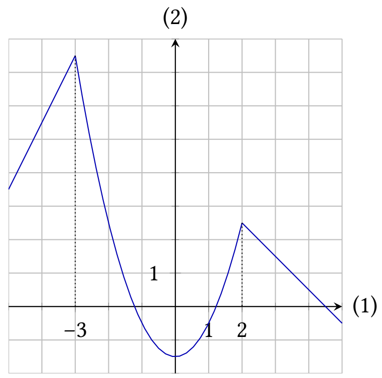


Figure 2.5: This function is not differentiable at $x = -3$ og $x = 2$; but it is continuous everywhere.

The graph has to be smooth for it to be differentiable because if there are kinks, we will get different tangent slopes depending on whether we approach the point from the left or from the right. Therefore there would not be a well-defined tangent slope.

2.2 Terms and notation

Definition 2.3 shows how to find the derivative $f'(x_0)$ of a function f at $x = x_0$. This number describes the tangent slope at the point where $x = x_0$. These tangent slopes then define a new function $f'(x)$, which has the tangent slopes at each point on the graph as function values. This function is also called the *derivative* of f.

To find the derivative, we look at the *difference quotient*[3] $\frac{\Delta f}{\Delta x} = \frac{f(x)-f(x_0)}{x-x_0}$. We investigate what happens to this quantity as x approaches x_0.

[3] $\frac{\Delta f}{\Delta x}$ is called the *difference quotient* because Δf and Δx are both differences, and the result of a division is called a quotient.

Because we let x approach x_0 in the difference quotient, the result is sometimes also referred to as the *differential quotient*. The term *derivative* is used also as a name for the function f', but the *differential quotient* is used only for the function value of $f'(x)$ at a specific point.

Because the derivative is found from the difference quotient $\frac{\Delta f}{\Delta x}$, we sometimes use the notation $\frac{\mathrm{d}f}{\mathrm{d}x}$ for the derivative.[4]

[4] Note that $\frac{\mathrm{d}f}{\mathrm{d}x}$ means exactly the same as $f'(x)$. This means that the symbol $\frac{\mathrm{d}f}{\mathrm{d}x}$ should *never* be understood as a fraction; it is not possible to separate $\mathrm{d}f$ and $\mathrm{d}x$.

So, the following statements are equivalent:

1. The derivative of $f(x) = 3x^2 + 7$ is $f'(x) = 6x$.
2. The derivative of $f(x) = 3x^2 + 7$ is $\frac{\mathrm{d}f}{\mathrm{d}x} = 6x$.

And the following statements are also equivalent:

1. The differential quotient of $f(x) = 3x^2 + 7$ at $x = 2$ is $f'(2) = 12$.
2. The differential quotient of $f(x) = 3x^2 + 7$ at $x = 2$ is $\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=2} = 12$.

2.3 Miscellaneous derivatives

In this section, we find the derivatives for a number of simple functions.

Theorem 2.5

The derivative of $f(x) = k$, where k is a constant, is $f'(x_0) = 0$.

This result follows from the fact that the graph of $f(x)$ is a line parallel to the x-axis, i.e. a line with slope 0. Since $f'(x_0)$ is the slope of the tangent at the point $(x_0, f(x_0))$, and the slope of the graph is 0 everywhere, $f'(x_0) = 0$. However, we also include a formal proof using definition 2.3:

Proof
If $f(x) = k$, then

$$\Delta f = f(x) - f(x_0) = k - k = 0\ .$$

Therefore

$$\frac{\Delta f}{\Delta x} = \frac{0}{x - x_0} = 0\ .$$

Since $\frac{\Delta f}{\Delta x} = 0$ for any value of Δx, we will also have

$$\frac{\Delta f}{\Delta x} \longrightarrow 0 \,, \qquad \text{as } x \longrightarrow x_0 \,.$$

So,

$$f'(x_0) = 0 \,.$$

■

Theorem 2.6

If $f(x) = x$, then $f'(x_0) = 1$.

The graph of $f(x) = x$ is a straight line with slope 1. The theorem follows from this fact. A formal proof using definition 2.3 is left as an exercise for the reader.

Theorem 2.7

When $f(x) = x^2$, the derivative is $f'(x_0) = 2x_0$.

Proof

First, we calculate

$$\Delta f = f(x) - f(x_0) = x^2 - x_0^2 = (x + x_0)(x - x_0) \,.$$

Next, we calculate the fraction $\frac{\Delta f}{\Delta x}$

$$\frac{\Delta f}{\Delta x} = \frac{(x + x_0)(x - x_0)}{x - x_0} = x + x_0 \,.$$

If $x \longrightarrow x_0$, this expression will approach $2x_0$.

Therefore $f'(x_0) = 2x_0$. ■

Theorem 2.8

When $f(x) = \frac{1}{x}$, the derivative is $f'(x_0) = -\frac{1}{x_0^2}$.

Proof

For $f(x) = \frac{1}{x}$, we have

$$\begin{aligned} \Delta f &= f(x) - f(x_0) \\ &= \frac{1}{x} - \frac{1}{x_0} \\ &= \frac{x_0}{x \cdot x_0} - \frac{x}{x \cdot x_0} \\ &= \frac{-(x - x_0)}{x \cdot x_0} \,. \end{aligned}$$

So,

$$\frac{\Delta f}{\Delta x} = \frac{\frac{-(x - x_0)}{x \cdot x_0}}{x - x_0} = \frac{-1}{x \cdot x_0} \,.$$

When $x \longrightarrow x_0$, this expression will approach $\frac{-1}{x_0 \cdot x_0} = \frac{-1}{x_0^2}$, and therefore

$$f'(x_0) = -\frac{1}{x_0^2} \,.$$

■

Theorem 2.9

If $f(x) = \sqrt{x}$, the derivative is $f'(x_0) = \frac{1}{2\sqrt{x_0}}$.

Proof

When $f(x) = \sqrt{x}$, we have

$$\Delta f = f(x) - f(x_0) = \sqrt{x} - \sqrt{x_0}\ .$$

We cannot rewrite this expression directly, so instead we try to rewrite the difference quotient $\frac{\Delta f}{\Delta x}$. Here, we can use a clever trick:[5]

[5]We multiply by $\sqrt{x} + \sqrt{x_0}$ in the numerator and the denominator. We can then use the rule

$$(a-b)(a+b) = a^2 - b^2\ .$$

$$\begin{aligned}\frac{\Delta f}{\Delta x} &= \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} \\ &= \frac{(\sqrt{x} - \sqrt{x_0}) \cdot (\sqrt{x} + \sqrt{x_0})}{(x - x_0) \cdot (\sqrt{x} + \sqrt{x_0})} \\ &= \frac{(\sqrt{x})^2 - (\sqrt{x_0})^2}{(x - x_0) \cdot (\sqrt{x} + \sqrt{x_0})} \\ &= \frac{x - x_0}{(x - x_0) \cdot (\sqrt{x} + \sqrt{x_0})} \\ &= \frac{1}{\sqrt{x} + \sqrt{x_0}}\ .\end{aligned}$$

This expression has the limit $\lim_{x \longrightarrow x_0} \frac{1}{\sqrt{x} + \sqrt{x_0}} = \frac{1}{\sqrt{x_0} + \sqrt{x_0}}$, i.e.

$$f'(x_0) = \frac{1}{2\sqrt{x_0}}\ . \qquad \blacksquare$$

Table 2.6: Various functions and their derivatives.

$f(x)$	$f'(x)$
k	0
x	1
x^2	$2x$
x^3	$3x^2$
x^n	nx^{n-1}
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
e^x	e^x
e^{kx}	$k\,e^{kx}$
a^x	$\ln(a) \cdot a^x$
$\ln(x)$	$\frac{1}{x}$

Table 2.6 lists further examples of derivatives.

Example 2.10 According to theorem 2.9, the derivative of $f(x) = \sqrt{x}$ is given by $f'(x_0) = \frac{1}{2\sqrt{x_0}}$. Since $f'(x)$ is the slope of the tangent to the graph, we can calculate that the tangent to the graph at the point $P(4, 2)$ has the slope

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2 \cdot 2} = \frac{1}{4}\ .$$

This is shown in figure 2.7.

So, the tangent is a straight line with slope $y = \frac{1}{4}x + b$. If we want to find the equation of this line, we can insert the point of tangency $P(4, 2)$ into the equation:

$$2 = \tfrac{1}{4} \cdot 4 + b \qquad \Longleftrightarrow \qquad b = 1\ .$$

Therefore, at the point $P(4, 2)$, the graph of $f(x) = \sqrt{x}$ has a tangent with the equation

$$y = \tfrac{1}{4}x + 1\ ,$$

which is also shown in figure 2.7.

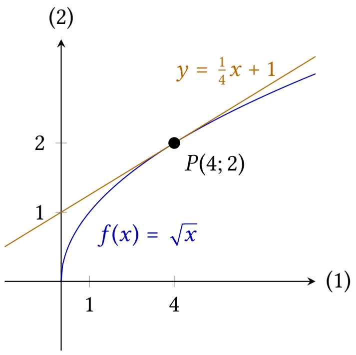


Figure 2.7: The graph of $f(x) = \sqrt{x}$ has a tangent with the equation $y = \frac{1}{4}x + 1$ at the point $P(4, 2)$.

2.4 Sum and difference

As it turns out, it is not necessary to use the method described in the previous sections, every time we want to find the derivative $f(x_0)$ of a given function f. It is sufficient to know the derivatives of a number of simple functions like the ones described above. Calculation rules exist, which we can use to find the derivative of a function f which is "built from" simpler functions.

Theorem 2.11

Let p be a differentiable function and c a constant, and let the function f be given by $f(x) = c \cdot p(x)$. Then

$$f'(x_0) = c \cdot p'(x_0) .$$

Proof
If $f(x) = c \cdot p(x)$, then

$$\begin{aligned}\Delta f &= c \cdot p(x) - c \cdot p(x_0) \\ &= c \cdot (p(x) - p(x_0)) = c \cdot \Delta p .\end{aligned}$$

I.e.

$$\frac{\Delta f}{\Delta x} = \frac{c \cdot \Delta p}{\Delta x} = c \cdot \frac{\Delta p}{\Delta x} .$$

If we let $x \longrightarrow x_0$, then $\frac{\Delta p}{\Delta x} \longrightarrow p'(x_0)$, and therefore $c \cdot \frac{\Delta p}{\Delta x} \longrightarrow c \cdot p'(x_0)$. So,

$$f'(x_0) = c \cdot p'(x_0) ,$$

which proves the theorem ■

This example demonstrates how to use the theorem:

Example 2.12 According to theorem 2.7, the derivative of $p(x) = x^2$ is given by $p'(x_0) = 2x_0$. But what is the derivative of $f(x) = 7x^2$?

Here, we can use theorem 2.11. If $f(x) = 7x^2$, then

$$f(x) = c \cdot p(x) , \qquad \text{where } c = 7 \text{ and } p(x) = x^2 .$$

Because we already know the derivative of $p(x) = x^2$, according to theorem 2.11, we get

$$f'(x_0) = c \cdot p'(x_0) = 7 \cdot 2x_0 = 14x_0 .$$

So, we can find the derivative of $f(x) = 7x^2$ quite easily, as long as we know the derivative of x^2.

Example 2.13 If we want to find the derivative of $f(x) = 4x^3$, we write $f(x)$ as $f(x) = 4 \cdot p(x)$, where $p(x) = x^3$.

The table of derivatives tells ud that $p'(x_0) = 3x_0^2$. According to theorem 2.11, we then have

$$f'(x_0) = 4 \cdot p'(x_0) = 4 \cdot 3x_0^2 = 12x_0^2 .$$

Theorem 2.14

Let p and q be two differentiable functions, and let $f(x) = p(x) + q(x)$. Then

$$f'(x_0) = p'(x_0) + q'(x_0) .$$

Proof

We use definition 2.3 and first calculate

$$\begin{aligned}\Delta f &= f(x) - f(x_0) = (p(x) + q(x)) - (p(x_0) - q(x_0)) \\ &= p(x) - p(x_0) + q(x) - q(x_0) \\ &= \Delta p + \Delta q .\end{aligned}$$

Then we get

$$\frac{\Delta f}{\Delta x} = \frac{\Delta p + \Delta q}{\Delta x} = \frac{\Delta p}{\Delta x} + \frac{\Delta q}{\Delta x} .$$

If we let $x \longrightarrow x_0$, then $\frac{\Delta p}{\Delta x} \longrightarrow p'(x_0)$ and $\frac{\Delta q}{\Delta x} \longrightarrow q'(x_0)$, which means that

$$f'(x_0) = p'(x_0) + q'(x_0) .$$ ■

Theorem 2.15

Let p and q be two differentiable functions, and let $f(x) = p(x) - q(x)$. Then

$$f'(x_0) = p'(x_0) - q'(x_0) .$$

This theorem can be proven in the exact same way as theorem 2.14. Therefore, the proof is left as an exercise for the reader.

Example 2.16 The theorems 2.11, 2.14 and 2.15 may be combined when we differentiate more complicated functions.

The function

$$f(x) = 4x^2 + 5\ln(x) - 3x$$

is a combination of the simpler functions x^2, $\ln(x)$, and x, which are all listed in table 2.6.

If we use theorem 2.14 and 2.15, we get

$$f'(x) = \left(4x^2\right)' + (5\ln(x))' - (3x)' .$$

We can then use theorem 2.11, so

$$f'(x) = 4 \cdot \left(x^2\right)' + 5 \cdot (\ln(x))' - 3 \cdot (x)' .$$

Next, we find the derivatives of x^2, $\ln(x)$, and x in the table. We then have

$$f'(x) = 4 \cdot 2x + 5 \cdot \frac{1}{x} - 3 \cdot 1 ,$$

which we can reduce to

$$f'(x) = 8x + \frac{5}{x} - 3 .$$

2.5 Products and inner linear functions

If we look at the theorems we have proven so far, we might get the impression that we can always differentiate complicated functions by simply differentiating their individual components. However, this is not the case, as the next theorem clearly shows.

Theorem 2.17: The product rule

Let p and q be two differentiable functions, and let $f(x) = p(x) \cdot q(x)$. Then

$$f'(x_0) = p'(x_0) \cdot q(x_0) + p(x_0) \cdot q'(x_0) \ .$$

Proof
When $f(x) = p(x) \cdot q(x)$,

$$\begin{aligned} \Delta f &= f(x) - f(x_0) \\ &= p(x) \cdot q(x) - p(x_0) \cdot q(x_0) \ . \end{aligned}$$

We want to rewrite this expression, so that it contains both Δp and Δq. To this end, we use a trick: We subtract the term $p(x_0) \cdot q(x)$ and then add it again. This does not change the expression:

$$\begin{aligned} \Delta f &= p(x) \cdot q(x) - p(x_0) \cdot q(x_0) \\ &= p(x) \cdot q(x) \underbrace{- p(x_0) \cdot q(x) + p(x_0) \cdot q(x)}_{\text{the sum of these two terms is 0}} - p(x_0) \cdot q(x_0) \ . \end{aligned}$$

Next, we can factor the expression like this:

$$\begin{aligned} \Delta f &= (p(x) - p(x_0)) \cdot q(x) + p(x_0) \cdot (q(x) - q(x_0)) \\ &= \Delta p \cdot q(x) + p(x_0) \cdot \Delta q \ . \end{aligned}$$

Then we have

$$\frac{\Delta f}{\Delta x} = \frac{\Delta p \cdot q(x) + p(x_0) \cdot \Delta q}{\Delta x} = \frac{\Delta p}{\Delta x} \cdot q(x) + p(x_0) \cdot \frac{\Delta q}{\Delta x} \ .$$

If we now let $\Delta x \longrightarrow 0$,

$$\begin{aligned} \frac{\Delta p}{\Delta x} &\longrightarrow p'(x_0) \\ q(x) &\longrightarrow q(x_0) \\ p(x_0) &\longrightarrow p(x_0) \\ \frac{\Delta q}{\Delta x} &\longrightarrow q'(x_0) \ . \end{aligned}$$

In total, we then get

$$\lim_{x \to x_0} \frac{\Delta f}{\Delta x} = p'(x_0) \cdot q(x_0) + p(x_0) \cdot q'(x_0) \ ,$$

and therefore

$$f'(x_0) = p'(x_0) \cdot q(x_0) + p(x_0) \cdot q'(x_0) \ .$$

■

Example 2.18 If we want to find the derivative of $f(x) = \sqrt{x} \cdot \ln(x)$, we write $f(x)$ as $f(x) = p(x) \cdot q(x)$, where

$$p(x) = \sqrt{x}, \qquad q(x) = \ln(x).$$

Looking at a table of derivatives, we get

$$p'(x) = \frac{1}{2\sqrt{x}}, \qquad q'(x) = \frac{1}{x}.$$

Theorem 2.17 then implies

$$\begin{aligned} f'(x) &= p'(x) \cdot q(x) + p(x) \cdot q'(x) \\ &= \frac{1}{2\sqrt{x}} \cdot \ln(x) + \sqrt{x} \cdot \frac{1}{x} . \end{aligned}$$

We can reduce this further:

$$f'(x) = \frac{\ln(x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \quad \Rightarrow \quad f'(x) = \frac{\ln(x) + 2}{2\sqrt{x}} .$$

In many different contexts, we see functions which are composed of a simple function and an inner linear function. This might be functions like

$$f(x) = \sqrt{2x - 3} \quad \text{and} \quad g(x) = e^{3x+10} .$$

In these cases, we can find the derivative in the following way:

Theorem 2.19

Let p be a differentiable function, and let $f(x) = p(ax + b)$, where a and b are two constants. Then

$$f'(x_0) = a \cdot p'(ax_0 + b) .$$

We will not prove this theorem, instead we provide some examples of how it is used:

Example 2.20 What is the derivative of $f(x) = \ln(7x - 5)$. This function can be written as $p(ax + b)$, where $p(t) = \ln(t)$; the derivative of $\ln(t)$ is $\frac{1}{t}$.

According to theorem 2.19, the derivative of f is then

$$f'(x) = 7 \cdot \frac{1}{7x - 5} = \frac{7}{7x - 5} .$$

Example 2.21 The derivative of $g(x) = \sqrt{8x + 3}$ is

$$g'(x) = 8 \cdot \frac{1}{2\sqrt{8x + 3}} ,$$

which reduces to

$$g'(x) = \frac{8}{2\sqrt{8x + 3}} = \frac{4}{\sqrt{8x + 3}} .$$

The derivative of $h(x) = 5 \cdot (2x - 1)^3$ is

$$h'(x) = 2 \cdot 5 \cdot 3 \cdot (2x - 1)^2 = 30 \cdot (2x - 1)^2 .$$

2.6 Composite functions and quotients

Theorem 2.19 deals with a special case of *composite functions*, i.e. functions like

$$f(x) = (\ln(x))^2 , \qquad g(x) = \sqrt{x^3 + 4} ,$$
$$h(x) = e^{6x+x^2} , \qquad k(x) = \ln(x^2 + e^x) .$$

The function f is a composite function because we can write it as $f = p \circ q$,[6] where the two functions p and q are the functions

[6]Remember that $f = p \circ q$ means that $f(x) = p(q(x))$ for all x.

$$p(q) = q^2 \quad \text{and} \quad q(x) = \ln(x) ,$$

We call p the *outer function* and q the *inner function*.

When we want to differentiate functions such as these, we can use the following theorem:

Theorem 2.22: The chain rule

Let p and q be differentiable functions, and let $f = p \circ q$. Then

$$f' = (p' \circ q) \cdot q' .$$

When we read this theorem, we need to remember that $f' = (p' \circ q) \cdot q'$ is a short-form notation, which means that

$$f'(x) = p'(q(x)) \cdot q'(x)$$

for all values of x.

Proof
If $f(x_0) = (p \circ q)(x_0)$, then

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0} = \frac{(p \circ q)(x) - (p \circ q)(x_0)}{x - x_0} .$$

Because $(p \circ q)(x) = p(q(x))$, we may write this as

$$\frac{\Delta f}{\Delta x} = \frac{p(q(x)) - p(q(x_0))}{x - x_0} = \frac{p(q(x)) - p(q(x_0))}{q(x) - q(x_0)} \cdot \frac{q(x) - q(x_0)}{x - x_0} .$$

Now we let $q = q(x)$ and $q_0 = q(x_0)$, so we can write this as

$$\frac{\Delta f}{\Delta x} = \frac{p(q) - p(q_0)}{q - q_0} \cdot \frac{\Delta q}{\Delta x} . \tag{2.1}$$

We then examine the two factors on the right hand side independently.

The fraction $\frac{p(q)-p(q_0)}{q-q_0}$ may be written as $\frac{\Delta p}{\Delta q}$, where it is implied that p is a function of q. I.e.

$$\frac{p(q) - p(q_0)}{q - q_0} \longrightarrow p'(q_0) , \qquad \text{as } x \longrightarrow x_0 .$$

For the fraction $\frac{\Delta q}{\Delta x}$, we have

$$\frac{\Delta q}{\Delta x} \longrightarrow q'(x_0)\ , \qquad \text{as } x \longrightarrow x_0\ .$$

In total, we get from equation (2.1) that

$$\frac{\Delta f}{\Delta x} \longrightarrow p'(q_0) \cdot q'(x_0)\ , \qquad \text{as } x \longrightarrow x_0\ .$$

Now, we remember that $q_0 = q(x_0)$, which means we can write this as $p'(q(x_0)) \cdot q'(x_0)$, i.e.

$$f' = (p' \circ q) \cdot q'\ .$$

■

Example 2.23 A function f is given by the formula $f(x) = \sqrt{x^2+3}$. Thus, f can be written as $f(x) = (p \circ q)(x)$, where

$$p(q) = \sqrt{q} \quad \text{og} \quad q(x) = x^2 + 3\ .$$

We differentiate these two functions:

$$p'(q) = \frac{1}{2\sqrt{q}} \quad \text{and} \quad q'(x) = 2x\ .$$

Theorem 2.22 then gives us

$$\begin{aligned} f'(x) &= p'(q) \cdot q'(x) \\ &= \frac{1}{2\sqrt{q}} \cdot 2x \\ &\overset{(*)}{=} \frac{1}{2\sqrt{x^2+3}} \cdot 2x \end{aligned}$$

At $(*)$ we replace q by $x^2 + 3$ because $q(x) = x^2 + 3$.

We can reduce this expression further to get

$$f'(x) = \frac{1}{2\sqrt{x^2+3}} \cdot 2x = \frac{x}{\sqrt{x^2+3}}\ .$$

Example 2.24 A function f is given by the formula $f(x) = \mathrm{e}^{x^2}$. To be able to differentiate f, we write this formula as $f(x) = (p \circ q)(x)$, where

$$p(q) = \mathrm{e}^q\ , \qquad q(x) = x^2\ .$$

A table of derivatives yields

$$p'(q) = \mathrm{e}^q\ , \qquad q'(x) = 2x\ .$$

Theorem 2.22 then gives us

$$f'(x) = p'(q) \cdot q'(x) = \mathrm{e}^q \cdot 2x = \mathrm{e}^{x^2} \cdot 2x\ .$$

The theorems 2.17 and 2.22 may also be used to prove a theorem about the derivative of a quotient of functions. We have the following theorem:

Theorem 2.25: The quotient rule

Let p and q be differentiable functions where $q(x) \neq 0$ for all x, and let $f = \frac{p}{q}$. Then

$$f' = \frac{p' \cdot q - p \cdot q'}{q^2} .$$

Proof
$f(x_0) = \left(\frac{p}{q}\right)(x_0)$ can be written as

$$f(x_0) = \frac{p(x_0)}{q(x_0)} = p(x_0) \cdot \frac{1}{q(x_0)} = p(x_0) \cdot \left(\frac{1}{q}\right)(x_0) .$$

This is a product of two functions, so according to theorem 2.17, we have

$$\begin{aligned} f'(x_0) &= p'(x_0) \cdot \left(\frac{1}{q}\right)(x_0) + p(x_0) \cdot \left(\frac{1}{q}\right)'(x_0) \\ &= \frac{p'(x_0)}{q(x_0)} + p(x_0) \cdot \left(\frac{1}{q}\right)'(x_0) . \end{aligned} \tag{2.2}$$

To continue this calculation, we need to investigate $\left(\frac{1}{q}\right)'(x_0)$. This is the derivative of a composite function. Using theorem 2.22, we get[7]

$$\left(\frac{1}{q}\right)'(x_0) = -\frac{1}{q(x_0)^2} \cdot q'(x_0) .$$

If we insert this result into (2.2), we get

$$\begin{aligned} f'(x_0) &= \frac{p'(x_0)}{q(x_0)} + p(x_0) \cdot \left(-\frac{1}{q(x_0)^2} \cdot q'(x_0)\right) \\ &= \frac{p'(x_0)}{q(x_0)} - \frac{p(x_0) \cdot q'(x_0)}{q(x_0)^2} \\ &= \frac{p'(x_0) \cdot q(x_0)}{q(x_0)^2} - \frac{p(x_0) \cdot q'(x_0)}{q(x_0)^2} \\ &= \frac{p'(x_0) \cdot q(x_0) - p(x_0) \cdot q'(x_0)}{q(x_0)^2} , \end{aligned}$$

So, $f' = \frac{p' \cdot q - p \cdot q'}{q^2}$. ■

Example 2.26 Let $f(x) = \frac{x^2}{e^x}$. We find the derivative $f'(x)$ by writing $f(x) = \left(\frac{p}{q}\right)(x)$, where

$$p(x) = x^2 , \qquad q(x) = e^x .$$

Then

$$p'(x) = 2x , \qquad q'(x) = e^x .$$

Now, theorem 2.25 gives us

$$f'(x) = \frac{p'(x) \cdot q(x) - p(x) \cdot q'(x)}{(q(x))^2}$$

[7] The function $\left(\frac{1}{q}\right)$ is a composite of $s(q) = \frac{1}{q}$ and $q(x)$. We then use that

$$s'(q) = -\frac{1}{q^2} .$$

$$= \frac{2x \cdot e^x - x^2 \cdot e^x}{(e^x)^2} \ .$$

We can reduce this further and find

$$f'(x) = \frac{2x - x^2}{e^x} \ .$$

Functions exist, where we cannot use just one of the theorems 2.17, 2.22 and 2.25. Sometimes we need to combine them.

Here, we show an elaborate example:

Example 2.27 A function is given by the formula

$$f(x) = \frac{1}{\sqrt{x^2 \cdot \ln(x)}} \ , \qquad x > 1 \ .$$

How do we differentiate this function?

First, we write $f(x) = (p \circ q)(x)$ where

$$p(q) = \frac{1}{q}, \qquad q(x) = \sqrt{x^2 \cdot \ln(x)} \ .$$

Here, we can easily differentiate $p(q)$, but what about $q(x)$? We split this further: $q(x) = (s \circ t)(x)$ where

$$s(t) = \sqrt{t} \ , \qquad t(x) = x^2 \cdot \ln(x) \ .$$

Now, we need to differentiate t. We do this by writing t as $t(x) = (n \cdot m)(x)$,

$$n(x) = x^2 \ , \qquad m(x) = \ln(x) \ .$$

Here,

$$n'(x) = 2x \ , \qquad m'(x) = \frac{1}{x} \ .$$

According to theorem 2.17, we then get

$$t'(x) = n'(x) \cdot m(x) + n(x) \cdot m'(x) = 2x \cdot \ln(x) + x^2 \cdot \frac{1}{x} \ .$$

Next, we reduce this to $t'(x) = 2x \cdot \ln(x) + x$.

Now, we have everything we need, and we can begin to work our way backwards through the many parts of the function:

$$q'(x) = (s' \circ t)(x) \cdot t'(x) = \frac{1}{2\sqrt{t}} \cdot (2x \cdot \ln(x) + x) = \frac{1}{2\sqrt{x^2 \cdot \ln(x)}} \cdot (2x \cdot \ln(x) + x) \ .$$

This reduces to

$$q'(x) = \frac{2x \cdot \ln(x) + x}{2\sqrt{x^2 \cdot \ln(x)}} \ .$$

Lastly, we therefore find

$$f'(x) = (p' \circ q)(x) \cdot q'(x) = -\frac{1}{q^2} \cdot \frac{2x \cdot \ln(x) + x}{2\sqrt{x^2 \cdot \ln(x)}}$$

$$= -\frac{1}{\left(\sqrt{x^2 \cdot \ln(x)}\right)^2} \cdot \frac{2x \cdot \ln(x) + x}{2\sqrt{x^2 \cdot \ln(x)}} .$$

This then reduces to

$$f'(x) = -\frac{2\ln(x) + 1}{2x^2 \cdot \ln(x) \cdot \sqrt{\ln(x)}} .$$

Using the theorems 2.11–2.25 and a table of derivatives allows us to differentiate any function. We therefore conclude this section by collecting these theorems:

Theorem 2.28

We can use the following rules to determine a derivative:

$$\begin{aligned}
f &= c \cdot p && \Rightarrow \quad f' = c \cdot p' . \\
f &= p + q && \Rightarrow \quad f' = p' + q' . \\
f &= p - q && \Rightarrow \quad f' = p' - q' . \\
f &= p \cdot q && \Rightarrow \quad f' = p' \cdot q + p \cdot q' . \\
f &= p \circ q && \Rightarrow \quad f' = (p' \circ q) \cdot q' . \\
f &= \frac{p}{q} && \Rightarrow \quad f' = \frac{p' \cdot q - p \cdot q'}{q^2} .
\end{aligned}$$

2.7 Trigonometric functions

The trigonometric functions sin, cos, and tan are also differentiable. When we treat sin and cos as mathematical functions, we need to remember that x is always measured in radians.

For sin and cos, we have the following:

Theorem 2.29

For the trigonometric functions sin and cos we have

1. If $f(x) = \sin(x)$, then $f'(x_0) = \cos(x_0)$.
2. If $f(x) = \cos(x)$, then $f'(x_0) = -\sin(x_0)$.

So, the functions sin and cos are almost each other's derivatives. Note, however, that we get a minus sign when we differentiate cos.

To prove the theorem, we look at the unit circle because the definitions of sin and cos are based on the unit circle. Here, we only prove the first part of the theorem.

Proof
Figure 2.8 shows a section of the unit circle with an arbitrary length of arc x_0. $\sin(x_0)$ and $\cos(x_0)$ are marked on the axes. The large marked triangle has a hypotenuse of 1, and the horizontal leg will have the length $a = \cos(x_0)$.

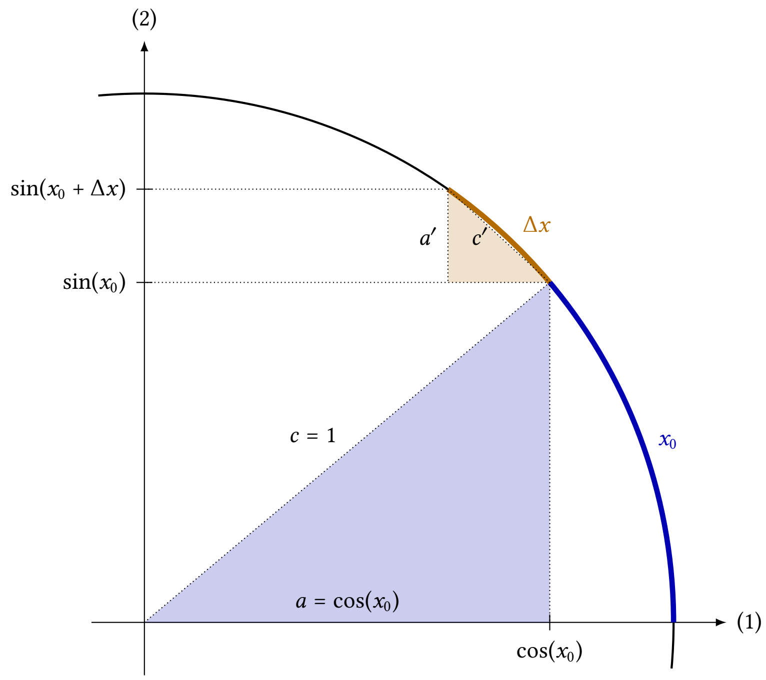


Figure 2.8: A section of the unit circle. The two marked triangles are almost similar when Δx is small. I.e. $\frac{a'}{c'} \approx \frac{a}{c}$.
When Δx is small, we also have $\Delta x \approx c'$.

If we add a small length of arc Δx, we can mark out a new triangle. Here, the hypotenuse is $c' \approx \Delta x$, as long as Δx is small. The vertical leg has a length of a'.

To determine $f'(x_0)$ when $f(x) = \sin(x)$, we first look at

$$\Delta f = f(x_0 + \Delta x) - f(x_0) = \sin(x_0 + \Delta x) - \sin(x_0) \ .$$

In the figure, we see that this corresponds to the line segment a', i.e. $\Delta f = a'$.

As stated, $\Delta x \approx c'$, and so

$$\frac{\Delta f}{\Delta x} = \frac{a'}{\Delta x} \approx \frac{a'}{c'} \ .$$

If Δx is small, the two hypotenuses are almost perpendicular. Then, the two marked triangles are almost similar, and we therefore have

$$\frac{a'}{c'} \approx \frac{a}{c} \ .$$

This means that

$$\frac{\Delta f}{\Delta x} \approx \frac{a}{c} \ ,$$

as long as Δx is small. And the smaller Δx is, the better the approximation.

If we then let $\Delta x \longrightarrow 0$, we get

$$f'(x_0) = \frac{a}{c} = \frac{\cos(x_0)}{1} = \cos(x_0) \ . \qquad \blacksquare$$

A corresponding geometric proof for the derivative of cos is left as an exercise for the reader.

Example 2.30 Figure 2.9 shows the graph of $f(x) = x + 2\sin(x)$. What is the derivative of f?

We know that the derivative of x is 1, and the derivative of $\sin(x)$ is $\cos(x)$ (according to theorem 2.29). Therefore,

$$f'(x) = 1 + 2\cos(x) \; .$$

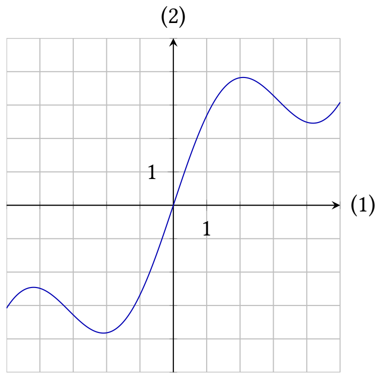


Figure 2.9: The graph of $f(x) = x + 2\sin(x)$.

Example 2.31 To find the derivative of $f(x) = 5 \cdot \sin(3x - 2)$, we need to use the rule from theorem 2.19. We can write f as

$$f(x) = p(3x - 2) \; ,$$

where $p(t) = 5 \cdot \sin(t)$, i.e. $p'(t) = 5 \cdot \cos(t)$

According to theorem 2.19, we then have

$$f'(x) = 3 \cdot 5 \cdot \cos(3x - 2) = 15 \cdot \cos(3x - 2) \; .$$

We can use theorem 2.29 and theorem 2.25 to prove the following theorem.

Theorem 2.32

The derivative of $f(x) = \tan(x)$ is $f'(x) = \frac{1}{\cos(x)^2}$.

Proof
From the definition of tan we have

$$f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)} \; .$$

According to theorem 2.25, the derivative is therefore

$$f'(x) = \frac{(\sin(x))' \cdot \cos(x) - \sin(x) \cdot (\cos(x))'}{\cos(x)^2} \; ,$$

and using theorem 2.29, we can write[8]

$$\begin{aligned} f'(x) &= \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot (-\sin(x))}{\cos(x)^2} \\ &= \frac{\cos(x)^2 + \sin(x)^2}{\cos(x)^2} \\ &= \frac{1}{\cos(x)^2} \; . \end{aligned}$$

■

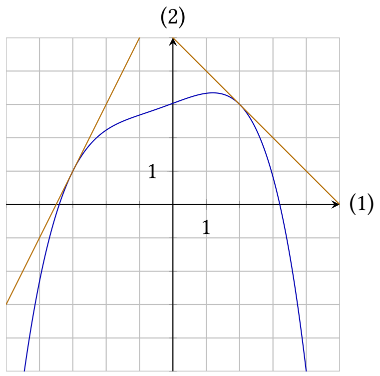
[8] In the calculation, we use that

$$\cos(x)^2 + \sin(x)^2 = 1 \; ,$$

which follows from the definition of cos and sin.

2.8 Exercises

Exercise 2.1
The figure shows the graph of a function f as well as the tangents to the graph of f at $x = -3$ and $x = 2$.



a) Use the graph to determine $f'(-3)$ and $f'(2)$.

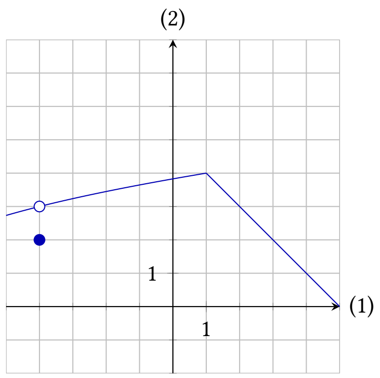
Exercise 2.2
Determine, by drawing the graphs, whether the following functions are differentiable at $x = 2$:

a) $f(x) = \begin{cases} 4x - 2 & \text{for } x < 2 \\ x^2 & \text{for } x \geq 2 \end{cases}$

b) $g(x) = |x - 2|$

Exercise 2.3
Determine the derivative $f'(x_0)$ of $f(x) = 2x^2 - x$ using the three-step method.

Exercise 2.4
The figure shows the graph of f. Find the values of x, for which this function is not differentiable.



Exercise 2.5
Use the three-step-method to determine the derivative $f'(x_0)$ of $f(x) = x^2 + 3x$.

Next, determine

a) $f'(1)$

b) $f'(-4)$

c) $\frac{df}{dx}$

d) $\left.\frac{df}{dx}\right|_{x=5}$

Exercise 2.6
Determine the derivatives of the following functions:

a) $f(x) = 4x - 1$

b) $g(x) = x^2 - x$

c) $h(x) = 5x - \sqrt{x}$

d) $k(x) = 3e^x - x^3 + 2x$

Exercise 2.7
Determine the value of the derivative $f'(x_0)$, when

a) $f(x) = 5x - \ln(x)$ and $x_0 = 1$

b) $f(x) = x^2 - 6\sqrt{x}$ and $x_0 = 9$

c) $f(x) = \frac{4}{x} + \frac{x^2}{4}$ and $x_0 = 2$

d) $f(x) = x^4 - 2x^3 + x^2 - 8x$ and $x_0 = 3$

e) $f(x) = \frac{\ln(x)}{4} - \frac{x^2}{6}$ and $x_0 = 1$

Exercise 2.8
Prove theorem 2.15.

Exercise 2.9
Differentiate the following functions:

a) $f(x) = x \cdot e^x$

b) $g(x) = x^2 \cdot \ln(x)$

c) $h(x) = 4x^2 \cdot \sqrt{x}$

d) $k(x) = 3x \cdot (e^x - 1)$

Exercise 2.10
Determine $\frac{df}{dx}$ when

a) $f(x) = 4\sqrt{x} \cdot \ln(x)$

b) $f(x) = \sqrt{x} \cdot (\sqrt{x} - 1)$

c) $f(x) = \frac{1}{x} \cdot \sqrt{x}$

d) $f(x) = (e^x + 1) \cdot (x^2 - 3)$

Exercise 2.11
Determine the derivatives of the following functions:

a) $f(x) = (3x - 1)^2$

b) $g(x) = e^{2x+9}$

c) $h(x) = \sqrt{7x - 5}$

d) $k(x) = 4 \cdot \ln(3 - x)$

Exercise 2.12
Determine at which values of x, the following functions have a tangent with slope 3.

a) $f(x) = x^2 - x$ b) $g(x) = \ln(x) - 2$

c) $h(x) = \sqrt{3x+1}$ d) $k(x) = \frac{3}{8-x}$

Exercise 2.13
The functions f and g are given by

$$f(x) = 2x - 1 \quad \text{and} \quad g(x) = 2\sqrt{x} - 3 .$$

Determine

a) $(fg)'(x)$ b) $(f\circ g)'(x)$ c) $(g\circ f)'(x)$

Exercise 2.14
Determine the derivatives of the following functions:

a) $f_1(x) = \sqrt{x^2 - 4}$ b) $f_2(x) = \ln(x^2 + 1)$

c) $f_3(x) = \ln(x)^3$ d) $f_4(x) = (e^x + 4)^2$

e) $f_5(x) = 4 \cdot \sqrt{\ln(x) - 3}$ f) $f_6(x) = \frac{1}{x^3+x^2-5}$

Exercise 2.15
Differentiate the following functions:

a) $f(x) = \dfrac{e^x}{4x}$ b) $g(x) = \dfrac{3x}{\ln(x)}$

c) $h(x) = \dfrac{x^2}{4+x}$ d) $k(x) = \dfrac{5-x^2}{\sqrt{x}}$

Exercise 2.16
Determine the derivatives of the following functions:

a) $f(x) = \dfrac{\sqrt{2x-1}}{3x}$

b) $g(x) = \ln\left(\dfrac{x-7}{x}\right)$

c) $h(x) = \sqrt{x} \cdot \dfrac{e^x}{x^2-1}$

d) $k(x) = \sqrt{\dfrac{x-3}{x^2+3}}$

Exercise 2.17
The graph of the function f passes through the point $(2, 3)$, and the graph of the function g passes through $(2, -1)$. Furthermore, $f'(2) = -4$ and $g'(2) = 5$.

Determine

a) $(3f)'(2)$ b) $(f+g)'(2)$

c) $(fg)'(2)$ d) $\left(\frac{f}{g}\right)'(2)$

Exercise 2.18
The graph of the function f has a tangent with slope 6 at the point $(2, 3)$, and the function g is given by the formula

$$g(x) = \ln(f(x) - 5) .$$

Determine $g'(2)$.

Exercise 2.19
Determine the derivative of each of these functions:

a) $f_1(x) = \sin(x) + \cos(x)$

b) $f_2(x) = \sin(x) \cdot \cos(x)$

c) $f_3(x) = \sin(x)^2$

d) $f_4(x) = \cos(4x - 5)$

e) $f_5(x) = x \cdot \sin(x)$

f) $f_6(x) = \dfrac{\sin(x)}{5-x}$

g) $f_7(x) = \sqrt{x} \cdot \sin(x)$

h) $f_8(x) = 4\cos(\sqrt{x} + 2)$

Exercise 2.20
The trigonometric function sec (secant) is defined by

$$\sec(x) = \frac{1}{\cos(x)} , \quad x \neq n\pi .$$

Show that

$$(\sec)'(x) = \frac{\tan(x)}{\cos(x)} \quad \text{and} \quad (\sec)'(x) = \sec(x) \cdot \tan(x) .$$

Exercise 2.21
Differentiate the following functions:

a) $f(x) = \sqrt{\sin(x)^2 + 1}$ b) $g(x) = \dfrac{\sin(x) - 1}{\cos(x)}$

c) $h(x) = \sin(x^3)$ d) $k(x) = \sin(\cos(x))$

Exercise 2.22
Determine the derivative of $f'\left(\frac{\pi}{2}\right)$ when

a) $f(x) = \dfrac{\cos(x) + 1}{\sin(x)}$

b) $f(x) = \sqrt{\cos(x) + 1}$

c) $f(x) = \sin(x)^3$

d) $f(x) = \cos(\sin(x) - 1)$

e) $f(x) = \dfrac{\sin(x) + 1}{\cos(x) + \sin(x)}$

3 Tangent equations

The value of the derivative is equal to the tangent slope at a given point on the graph. If we know the tangent slope and a point, we can determine an equation for the tangent. A few examples are given below.

Example 3.1 The function $f(x) = x^2 + 4x + 6$ has a tangent at the point $P(-1, f(-1))$. What is the equation of this tangent?

The tangent is a straight line, so it can be described by the equation $y = ax + b$. We therefore need to find the two numbers a and b to be able to write down the equation. a is the tangent slope, which is given by $f'(x)$, and therefore we begin by determining $f'(x)$:

$$f'(x) = 2x + 4 \cdot 1 + 0 = 2x + 4\ .$$

The x-coordinate of the point is $x_0 = -1$, so the tangent slope is

$$f'(-1) = 2 \cdot (-1) + 4 = 2\ ,$$

and the equation of the tangent is $y = 2x + b$.

To determine the whole equation, we need to know the point of tangency. The x-coordinate is $x_0 = -1$, and the y-coordinate is

$$y_0 = f(-1) = (-1)^2 + 4 \cdot (-1) + 6 = 1 - 4 + 6 = 3\ .$$

Thus the point of tangency is $(-1, 3)$. We insert this point into the equation of the tangent, i.e.

$$3 = 2 \cdot (-1) + b \qquad \Leftrightarrow \qquad b = 5\ .$$

So, the equation of the tangent is

$$y = 2x + 5\ .$$

Figure 3.1 shows the graph and the tangent.

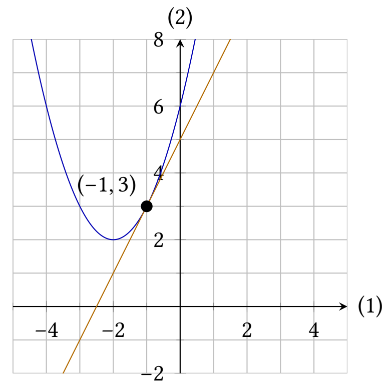


Figure 3.1: The graph of $f(x) = x^2 + 4x + 6$ has a tangent with equation $y = 2x + 5$ at the point $P(-1, 3)$.

Example 3.2 The function $g(x) = 3x + \ln(x)$ has a tangent at the point $P(1, f(1))$.

To determine an equation for the tangent, we first determine

$$g'(x) = 3 + \frac{1}{x}.$$

The slope of the tangent is then

$$a = f'(1) = 3 + \frac{1}{1} = 4 \,,$$

and the equation becomes $y = 4x + b$.

To determine b we calculate the y-coordinate of the point of tangency:

$$y_0 = f(1) = 3 \cdot 1 + \ln(1) = 3 \,,$$

and we insert this number as well as the x-coordinate $x_0 = 1$ into the equation of the tangent:

$$3 = 4 \cdot 1 + b \qquad \Longleftrightarrow \qquad b = -1.$$

Therefore, the equation of the tangent is

$$y = 4x - 1 \,.$$

As the two previous examples show, we use the same method every time we determine an equation of a tangent at a given point. This method can be turned into a formula, as the following theorem shows:

Theorem 3.3

Let a differentiable function $f(x)$ be given. Then the tangent to the graph of f at the point $P(x_0, f(x_0))$ has the equation

$$y = f'(x_0) \cdot (x - x_0) + f(x_0) \,.$$

Proof

The tangent is a straight line, so its equation is $y = ax + b$. Because the value of $f'(x)$ equals the tangent slope, and the point of tangency is $P(x_0, f(x_0))$, the slope of the tangent is

$$a = f'(x_0) \,.$$

So, we can write the equation of the tangent as

$$y = f'(x_0) \cdot x + b \,. \tag{3.1}$$

To determine the y-axis intercept b, we insert the known point[1] $P(x_0, f(x_0))$ into the equation of the tangent and solve for b:

$$f(x_0) = f'(x_0) \cdot x_0 + b \qquad \Longleftrightarrow \qquad b = -f'(x_0) \cdot x_0 + f(x_0) \,.$$

Next, we insert this expression for b into the tangent equation (3.1), and we get

$$y = f'(x_0) \cdot x - f'(x_0) \cdot x_0 + f(x_0) \,,$$

and by factoring, we get the equation

$$y = f'(x_0) \cdot (x - x_0) + f(x_0) \,. \qquad \blacksquare$$

Here, we provide a few examples on how to use the formula:

[1] Remember that the graph of f and the tangent both pass through $P(x_0, f(x_0))$, i.e. this point has to fit into the equation.

Example 3.4 The function $f(x) = 3x^2 + 10$ has a tangent at the point $P(5, f(5))$. To determine an equation for this tangent, we use the formula

$$y = f'(x_0) \cdot (x - x_0) + f(x_0)$$

with $x_0 = 5$, i.e.

$$y = f'(5) \cdot (x - 5) + f(5) \ .$$

Before we can use this formula, we need to know $f'(x)$:

$$f'(x) = 3 \cdot 2x + 0 = 6x \ .$$

Then we calculate

$$\begin{aligned} f'(5) &= 6 \cdot 5 = 30 \\ f(5) &= 3 \cdot 5^2 + 10 = 85 \ . \end{aligned}$$

Inserting these numbers into the formula, we get

$$y = 30 \cdot (x - 5) + 85 \ ,$$

which reduces to

$$y = 30x - 65 \ .$$

Example 3.5 The function $g(x) = (7x + 1) \cdot \mathrm{e}^x$ has a tangent at the point $P(0, g(0))$.

The tangent has the equation

$$y = g'(0) \cdot (x - 0) + g(0) = g'(0) \cdot x + g(0) \ .$$

We now find[2]

[2] We differentiate the function using the product rule, theorem 2.17.

$$g'(x) = 7 \cdot \mathrm{e}^x + (7x + 1) \cdot \mathrm{e}^x = (7x + 8) \cdot \mathrm{e}^x \ .$$

I.e.

$$\begin{aligned} g'(0) &= (7 \cdot 0 + 8) \cdot \mathrm{e}^0 = 8 \cdot 1 = 8 \\ g(0) &= (7 \cdot 0 + 1) \cdot \mathrm{e}^0 = 1 \cdot 1 = 1 \ . \end{aligned}$$

When we insert this into the expression above, we get the equation

$$y = 8x + 1 \ .$$

3.1 Determining points of tangency

If we know the formula of a function and a point on the graph, we can determine an equation of the tangent to the graph at this point. But it is equally possible to work backwards and find the point of tangency if we know an equaiton of the tangent.

In this section, we show a few examples:

Example 3.6 A function is given by the formula $f(x) = -x^2 + 3x + 1$.

The graph of the function has a tangent with equation $y = x + 2$. Where on the graph do we find the point of tangency of this tangent?

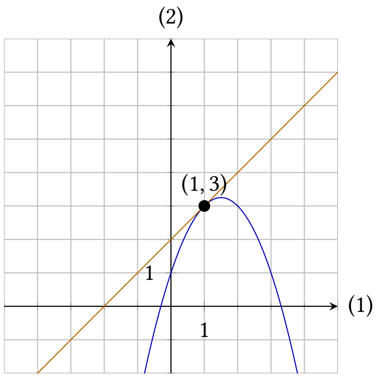


Figure 3.2: The tangent $y = x + 2$ touches the graph of $f(x) = -x^2 + 3x + 1$ at the point $(1, 3)$.

The derivative of f is

$$f'(x) = -2x + 3 ,$$

and the value of this function is equal to the tangent slope at each point on the graph.

The tangent, of which we know the equation, has slope 1, i.e. $f'(x) = 1$ at the point of tangency. This gives us the equation

$$-2x + 3 = 1 \qquad \Longleftrightarrow \qquad x = 1 .$$

So, the point of tangency has the x-coordinate 1. Now, we need to find the y-coordinate, which is

$$f(1) = -1^2 + 3 \cdot 1 + 1 = 3 .$$

Therefore, the point of tangency has the coordinates $(1, 3)$, see figure 3.2.

Example 3.7 The function f is given by

$$f(x) = x - \frac{4}{x} + 3 , \quad x > 0 .$$

The graph of f has a tangent with slope 2. Where is the point of tangency for this tangent, and what is its equation?

Since $f'(x)$ equals the tangent slope, we need to determine when $f'(x) = 2$. Therefore, we first find $f'(x)$,

$$f'(x) = 1 + \frac{4}{x^2} , \quad x > 0 .$$

Next, we solve the equation $f'(x) = 2$,

$$1 + \frac{4}{x^2} = 2 \qquad \Longleftrightarrow \qquad \frac{4}{x^2} = 1 \qquad \Longleftrightarrow \qquad x = -2 \ \vee \ x = 2 .$$

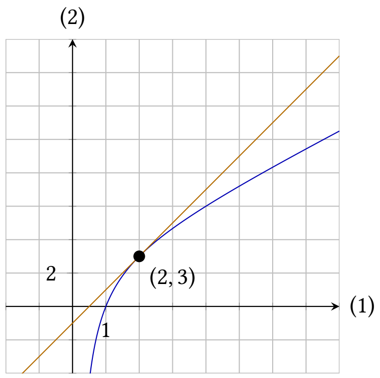


Figure 3.3: The graph of $f(x) = x - \frac{4}{x} + 3$ has a tangent with slope 2 at the point $(2, 3)$.

There are two solutions to this equation, but because $f(x)$ is only defined for $x > 0$, we discard the negative solution. So, the x-coordinate of the point of tangency is $x = 2$.

The y-coordinate of the point of tangency is

$$f(2) = 2 - \frac{4}{2} + 3 = 3 ,$$

and the point of tangency has the coordinates $(2, 3)$, see figure 3.3.

So, according to theorem 3.3, the equation of the tangent is given by

$$y = f'(2) \cdot (x - 2) + f(2) ,$$

but because we already know the slope of the tangent, $f'(2) = 2$, and have calculated $f(2) = 3$, this equation becomes

$$y = 2 \cdot (x - 2) + 3 ,$$

which reduces to

$$y = 2x - 1 .$$

Example 3.8 The graph of the function $f(x) = x^3 - 3x^2 - 21x + 5$ has two tangents with slope 3. What are the points of tangency for these tangents?

The slope of the tangents is 3, i.e. $f'(x) = 3$. To solve this equation, we first need to determine $f'(x)$,

$$f'(x) = 3x^2 - 3 \cdot 2x - 21 \cdot 1 = 3x^2 - 6x - 21 \ .$$

The equation $f'(x) = 3$ is the quadratic equation

$$3x^2 - 6x - 21 = 3 \qquad \Longleftrightarrow \qquad 3x^2 - 6x - 24 = 0 \ .$$

If we solve this equation, we find the two solutions

$$x = -2 \ \vee \ x = 4 \ .$$

So, the two points of tangency are $(-2, f(-2))$ og $(4, f(4))$. We can now determine the two y-coordinates

$$f(-2) = (-2)^3 - 3 \cdot (-2)^2 - 21 \cdot (-2) + 5 = 27$$
$$f(4) = 4^3 - 3 \cdot 4^2 - 21 \cdot 4 + 5 = -63 \ .$$

Therefore, the two points of tangency are $(-2, 27)$ and $(4, -63)$. At these two points, the graph of f has tangents with slope 3.

If we want to find the equations of these two tangents, we can use the same method as in example 3.7.

Example 3.9 In example 3.8, we saw that the graph of $f(x) = x^3 - 3x^2 - 21x + 5$ has two tangents with slope 3. Is there a slope a, so that the graph has exactly one tangent with this slope?

This question is more complicated, but because we find the point of tangency of the tangents by solving the equation $f'(x) = a$ for a specific slope a, we can rephrase the question as: Does a number a exist, so that the equation

$$f'(x) = a \tag{3.2}$$

has exactly one solution?

From example 3.8, we have

$$f'(x) = 3x^2 - 6x - 21 \ .$$

so equation (3.2) becomes

$$3x^2 - 6x - 21 = a \qquad \Longleftrightarrow \qquad 3x^2 - 6x - 21 - a = 0 \ .$$

This is a quadratic equation. If this equation is to have exactly one solution, its discriminant must be equal to 0. The discriminant of this equation is[3]

$$d = (-6)^2 - 4 \cdot 3 \cdot (-21 - a) = 36 - 12 \cdot (-21 - a) = 288 + 12a \ .$$

If this must equal 0, then

$$288 + 12a = 0 \qquad \Longleftrightarrow \qquad 12a = -288 \qquad \Longleftrightarrow a = -24 \ .$$

So, the graph has exactly one tangent with slope $a = -24$.

Actually, we can examine the discriminant further and find that when $a > -24$, the graph has two tangents with slope a, whereas the graph has no tangents with slope a when $a < -24$.

[3]Remember that the discriminant is $d = B^2 - 4AC$, where A, B and C are the coefficients of the equation. (We write A, B and C, because the coefficient of the second degree term cannot be denoted by a, since this is the tangent slope.)

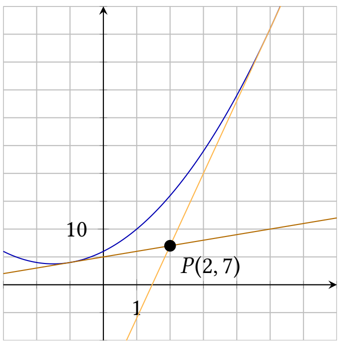


Figure 3.4: The graph of $f(x) = x^2 + 3x + 6$ has two tangents passing through $P(2,7)$.

Example 3.10 In this example, we look at the graph of the function $f(x) = x^2 + 3x + 6$. How many of the tangents to the graph also pass through the point $P(2,7)$?

Answering this question is quite complicated, because the point P is not on the graph. Figure 3.4 depicts this situation; here we see that two tangents to the graph of f pass through the point P.

According to theorem 3.3, the equation of the tangent is

$$y = f'(x_0) \cdot (x - x_0) + f(x_0) \ .$$

The problem is now to find the points of tangency for those tangents which pass through $P(2,7)$. We can find the points of tangency if we know the their x-coordinates, x_0, so we need to find these coordinates.

We know that the tangents pass through the point $P(2,7)$, so these coordinates have to fit into the equation of the tangent, i.e. we have

$$7 = f'(x_0) \cdot (2 - x_0) + f(x_0) \ . \tag{3.3}$$

To solve this equation, we need to know $f'(x)$, so we differentiate f:

$$f'(x) = 2x + 3 \ .$$

We insert this and the formula of the function itself into the equation (3.3), and we get

$$7 = (2x_0 + 3) \cdot (2 - x_0) + (x_0^2 + 3x_0 + 6) \ ,$$

which reduces to

$$7 = -2x_0^2 + x_0 + 6 + x_0^2 + 3x_0 + 6 \ ,$$

which we can reduce further to arrive at the quadratic equation

$$x_0^2 - 4x_0 - 5 = 0 \ .$$

This equation has the solution

$$x_0 = -1 \ \vee \ x_0 = 5 \ .$$

Because there are two points of tangency, there are two tangents. The y-coordinates of the points of tangency and the equations of the tangents can then be found by the same method as the one we used in example 3.4.

3.2 Exercises

Exercise 3.1
Determine an equation for the tangent to the graph of the given function at the given point:

a) $f(x) = x^2 + 1\ ,\quad (3, 10)$

b) $g(x) = 3x - x^2\ ,\quad (1, 2)$

c) $h(x) = 2\ln(x) + 5\ ,\quad (1, 5)$

Exercise 3.2
Determine an equation for the tangent to the graph of the given function at the given point:

a) $f(x) = x^3 + x^2 - 4\ ,\quad (1, f(1))$

b) $g(x) = \mathrm{e}^x - 4x\ ,\quad (0, g(0))$

c) $h(x) = 8\sqrt{x} + 3x\ ,\quad (4, h(4))$

Exercise 3.3
Determine an equation for the tangent to the graph of the given function at the given point:

a) $f(x) = \mathrm{e}^x \cdot (x^2 + 1)\ ,\quad (0, f(0))$

b) $g(x) = \sqrt{3x + 1}\ ,\quad (5, g(5))$

Exercise 3.4
The graph of the function $f(x) = x^2 + 5x$ has a tangent with slope 3.

Determine the point of tangency of this tangent.

Exercise 3.5
The graph of the function $g(x) = x^3 - x^2 + x + 4$ has two tangents with slope 1.

Determine the points of tangency of these tangents, and determine an equation for each of the tangents.

Exercise 3.6
The function f is given by the formula

$$f(x) = x^2 - 5x + 7\ .$$

The graph of the function has a tangent t at the point $P(2, f(2))$.

a) Determine an equation for t.

The graph has another tangent s which is perpendicular to t.

b) Determine the point of tangency for s.

c) Determine an equation for s.

Exercise 3.7
A tangent to the graph of $f(x) = \sqrt{2x + 10}$ has its point of tangency at $(x_0, 4)$.

a) Determine x_0.

b) Determine an equation for the tangent.

Exercise 3.8
The function f given by

$$f(x) = \frac{x - 1}{x^2 + 3}$$

has two horizontal tangents.

Determine the points of tangency for these two tangents.

Exercise 3.9
The function f is given by

$$f(x) = \frac{1}{1 - x}\ .$$

For two values of k, the line with equation $y = x + k$ is a tangent to the graph of f.

a) Determine the two values of k.

Exercise 3.10
The function f is given by

$$f(x) = x^3 + 3x^2 - 4x + 1\ .$$

The graph of f has exactly one tangent with slope a.

a) Determine the value of a.

Exercise 3.11
The function f is given by

$$f(x) = x^2 + 3x - 6\ .$$

The graph of f has two tangents passing through the point $P(2, -5)$, which is not on the graph.

a) Determine an equation for each of these tangents.

Exercise 3.12
A *normal line* is a line which is perpendicular to the tangent at the point of tangency.

Determine an equation for the normal line to the graph of

$$f(x) = \frac{1 - x}{x^2 + 1}$$

at the point $P(1, f(1))$.

4 Monotonicity and turning points

If a function behaves in such a way that its function values increase whenever the independent variable increases, we call the function *increasing*.

If, on the other hand, the function values decrease when the independent variable increases, the function is called *decreasing*.

Formally, we have the following definition:

Definition 4.1

Let a function f be defined in an interval.

1. If for any arbitrary pair of numbers x_1, x_2 in this interval we have
$$x_1 \leq x_2 \quad \Longrightarrow \quad f(x_1) \leq f(x_2)\ ,$$
the function is said to be *decreasing* in this interval.
2. If for any arbitrary pair of numbers x_1, x_2 in this interval we have
$$x_1 \leq x_2 \quad \Longrightarrow \quad f(x_1) \geq f(x_2)\ ,$$
the function is said to be *decreasing* in this interval.

Note that the definition concerns the functions behaviour in *intervals*. If we look only at a point, it does not make sense to talk about whether the function is increasing or decreasing. The properties *increasing* and *decreasing* therefore apply to intervals, not points.

Example 4.2 The graph of the function $f(x) = 2x + 1$ is a straight line with positive slope. Therefore, this function is increasing.

On the other hand, a straight line with a negative slope is decreasing (e.g. the function $f(x) = -4x + 3$.)

A function which is increasing or decreasing everywhere, is called a *monotonous* function. However, not all functions are monotonous. Many functions exist, which are increasing in some intervals and decreasing in others.

When we describe where a function is increasing and where it is decreasing, we describe the function's *properties of monotonicity*. We find the properties of monotonicity by dividing the x-axis into the intervals in which the function is increasing and those in which it is decreasing.

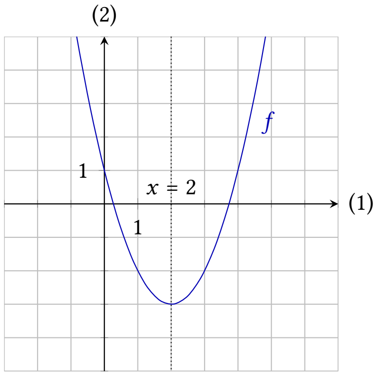


Figure 4.1: The graph of $f(x) = x^2 - 4x + 1$.

Example 4.3 Figure 4.1 shows the graph of the function

$$f(x) = x^2 - 4x + 1\ .$$

A vertical line is also drawn at $x = 2$. We see that the function is decreasing to the left side of the line and increasing to the right.

So, the properties of monotonicity of this function are that $f(x)$ is decreasing when $x \leq 2$, and increasing when $x \geq 2$.

In example 4.3, we determined the properties of monotonicity by looking at the graph. We can always draw the graph of a given function and determine the properties of monotonicity this way, but our precision will be limited.

Therefore, we need a method to calculate from the formula of the function where the graph changes from decreasing to increasing or vice versa.

From example 4.2, we have that when the graph of a function is a straight line, its properties of monotonicity is determined by the slope. If the slope is positive, the functions is increasing, and when it is negative, the function is decreasing.

The tangents of a function are straight lines, and their slopes are determined by $f'(x)$, wherefore the following theorem is intuitively true:

Theorem 4.4

When the function f is differentiable, we have:

1. If f is increasing in the interval $[a;b]$, then $f'(x) \geq 0$ for all $x \in\ ]a;b[$.
2. If f is decreasing in the interval $[a;b]$, then $f'(x) \leq 0$ for all $x \in\ ]a;b[$.
3. If f is constant in the interval $[a;b]$, then $f'(x) = 0$ for all $x \in\ ]a;b[$.

Note here that when $f(x)$ is increasing, the tangent slope is not necessarily strictly positive in the entire interval. It is allowed to be 0 in some places. This follows from definition 4.1, where we do not demand that $f(x_1)$ is greater than $f(x_2)$ when $x_1 \leq x_2$, but merely that it is greater than *or equal to*. So, increasing as well as decreasing functions can be constant in an interval.[1]

[1] Actually, according to definition 4.1, a constant function is simultaneously increasing and decreasing. This might seem contradictory, but it is true nonetheless.

Theorem 4.4 may be used to determine the properties of $f'(x)$ when we already know if the function is increasing or decreasing. Usually, we would instead like to determine the properties of monotonicity based on our knowledge of $f'(x)$. Here, we have the following theorem:

Theorem 4.5: Monotonisætningen

For a differentiable function f, we have:

1. If $f'(x) > 0$ for all x in an interval $]a; b[$, then f is increasing in $[a; b]$.
2. If $f'(x) < 0$ for all x in an interval $]a; b[$, then f is decreasing in $[a; b]$.
3. If $f'(x) = 0$ for all x in an interval $]a; b[$, then f is constant in $[a; b]$.

So, if we want to determine the properties of monotonicity of a function f, we need to investigate f' to determine when the value of $f'(x)$ changes from positive to negative or vice versa.

If $f'(x)$ changes from positive to negative, the value of $f'(x)$ must pass through 0. So, we need to find out when $f'(x) = 0$. This is illustrated in the following example:

Example 4.6 Here, we look again at the function from example 4.3,

$$f(x) = x^2 - 4x + 1 \ .$$

To find out when the graph changes from increasing to decreasing, we need to find out where $f'(x) = 0$. First, we therefore determine $f'(x)$,

$$f'(x) = 2x - 4 \ .$$

The equation $f'(x) = 0$ then becomes

$$2x - 4 = 0 \qquad \Longleftrightarrow \qquad x = 2 \ .$$

At $x = 2$ the graph has a tangent with slope 0, i.e. a horizontal tanget. This is also shown in figure 4.2.

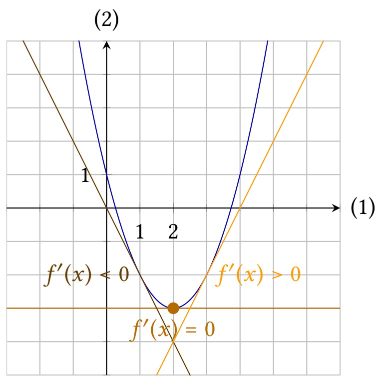


Figure 4.2: The graph of $f(x) = x^2 - 4x + 1$ is decreasing before $x = 2$ and increasing after $x = 2$. At $x = 2$, we have a horizontal tangent.

On the graph, we see that the function is decreasing before $x = 2$ and increasing after $x = 2$. If we do not have the graph, we need to determine the sign of $f'(x)$ by calculation.

If we want to know whether $f'(x)$ is positive or negative when $x < 2$, we choose a number less than 2, which we insert into the formula for f'. A number less than 2 might be e.g. 0. Then we get

$$f'(0) = 2 \cdot 0 - 4 = -4 \ .$$

Because $-4 < 0$, we know that $f'(x)$ is negative for all $x < 2$, i.e. f is decreasing in this interval.[2]

We also choose a number greater than 2, e.g. 3, and calculate

$$f'(3) = 2 \cdot 3 - 4 = 2 > 0 \ .$$

So, $f'(x)$ is positive for $x > 2$, and therefore $f(x)$ is increasing for these values of x.

In total, the properties of monotonicity of f are that $f(x)$ is decreasing for $x \leq 2$ and increasing for $x \geq 2$.[3]

[2]We know that $f'(x)$ is only 0 when $x = 2$. Therefore, the value of $f'(x)$ will have the same sign for all numbers $x < 2$, and it is only necessary to investigate the sign of $f'(x)$ for one number less than 2; here we chose $x = 0$.

[3]The numbers 0 and 3, which we used to determine the sign of $f'(x)$ are not part of the properties of monotonicity. They are merely two arbitrary numbers less than and greater than 2, which we used to determine the sign of $f'(x)$ when x is less than/greater than 2.

4.1 Sign table

A *sign table* is a tool often used to sum up calculations, before we present the properties of monotonicity. For the function in example 4.6, a sign table could look like this:

x :		2		→
$f'(x)$:	$-$	0	$+$	
$f(x)$:	↘	min.	↗	

This figure shows that before $x = 2$, $f'(x) < 0$, and after $x = 2$, $f'(x) > 0$. This is shown by the $-$ and $+$ signs in the figure. In the last line, we see that this corresponds to $f(x)$ being decreasing and increasing (illustrated by the two arrows ↘ and ↗).

Using the sign table, we can easily present the properties of monotonicity.[4] But we can also see something else. At $x = 2$ the function f has a *minimum*, i.e. a point on the graph where the function value has its lowest possible value.

[4]Note that the sign table is not the same as the properties of monotonicity, but that the properties of monotonicity may be read from the sign table.

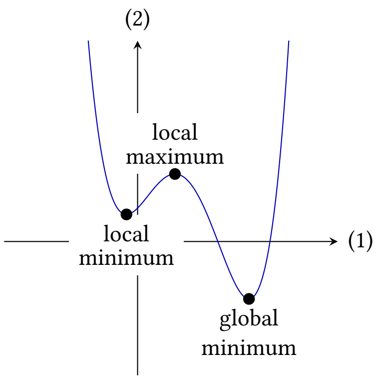


Figure 4.3: Functions may have global as well as local turning points.

We see from the table that it is a minimum because the function is first decreasing and then increasing. In this case, it is actually a *global* minimum because it is the lowest point on the entire graph. A minimum which is not global is called a *local* minimum. In the same way, we refer to global and local maxima. Figure 4.3 illustrates this.

A collective term for these points on the graph is *turning points* (or *extrema*). So, a turning point is a point on the graph which is either a maximum or a minimum (local or global).

Example 4.7 In this example, we determine the properties of monotonicity and the turning points of the function $f(x) = x^3 - 6x^2 + 9x + 1$.

The derivative is

$$f'(x) = 3x^2 - 12x + 9 ,$$

i.e. the equation $f'(x) = 0$ is the quadratic equation

$$3x^2 - 12x + 9 = 0 ,$$

which has the solutions $x = 1$ and $x = 3$.

These two solutions divides the number line into three intervals: The numbers less than 1, the numbers between 1 and 3, and the numbers greater than 3. Now, we choose a number from each of these intervals to determine the signs of $f'(x)$ in each of them:

$$\begin{aligned} x < 1 :&\quad f'(0) = 3 \cdot 0^2 - 12 \cdot 0 + 9 = 9 > 0 \\ 1 < x < 3 :&\quad f'(2) = 3 \cdot 2^2 - 12 \cdot 2 + 9 = -3 < 0 \\ x > 3 :&\quad f'(5) = 3 \cdot 5^2 - 12 \cdot 5 + 9 = 24 > 0 \end{aligned}$$

This allows us to write a sign table:

x :		1		3	
$f'(x)$:	+	0	−	0	+
$f(x)$:	↗	max.	↘	min.	↗

From this sign table we find the properties of monotonicity:

> $f(x)$ is increasing for $x \leq 1$ and for $x \geq 3$, and decreasing for $1 \leq x \leq 3$.

Because we know that the intervals of monotonicity are separated at $x = 1$ and $x = 3$, we can also find the properties of monotonicity by looking at the graph (see figure 4.4) instead of writing the sign table.

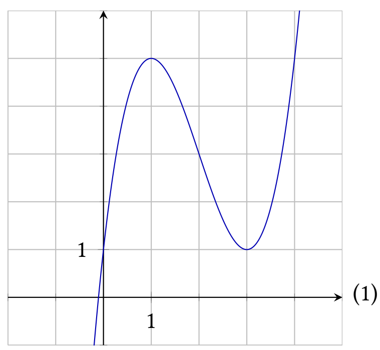


Figure 4.4: The graph of $f(x) = x^3 - 6x^2 + 9x + 1$ has a local maximum and a local minimum.

From the sign table, we can also see that there are two local turning points. One of them is a local maximum at $x = 1$, the other is a local minimum at $x = 3$.

We find the y-coordinates of the two turning points,

$$f(1) = 1^3 - 6 \cdot 1^2 + 9 \cdot 1 + 1 = 5$$
$$f(3) = 3^3 - 6 \cdot 3^2 + 9 \cdot 3 + 1 = 1\ .$$

So, the function f has a local maximum at $(1, 5)$ and a local minimum at $(3, 1)$.

Example 4.8 In this example, we determine possible turning points of the function

$$f(x) = 6 \cdot \sqrt{x} - 2x\ , \qquad x > 0\ .$$

The graph of this function is shown in figure 4.5. We see that the function appears to have a global maximum near $x = 2$.

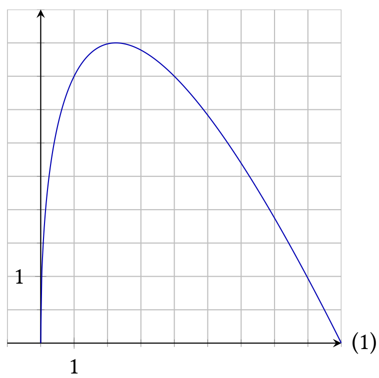


Figure 4.5: The graph of $f(x) = 6 \cdot \sqrt{x} - 2x$ appears to have a global maximum.

To determine whether the function has a global maximum, we first determine

$$f'(x) = 6 \cdot \frac{1}{2 \cdot \sqrt{x}} - 2 \cdot 1 = \frac{3}{\sqrt{x}} - 2\ .$$

The equation $f'(x) = 0$ then becomes

$$\frac{3}{\sqrt{x}} - 2 = 0 \qquad \Longleftrightarrow \qquad 2\sqrt{x} = 3 \qquad \Longleftrightarrow \qquad x = \left(\frac{3}{2}\right)^2 = \frac{9}{4}\ .$$

So, there is a possible turning point at $x = \frac{9}{4}$.

We want to write a sign table, so we look at $f'(x)$ for $x < \frac{9}{4}$ and for $x > \frac{9}{4}$.[5]

$$0 < x < \tfrac{9}{4}\ : \quad f'(1) = \frac{3}{\sqrt{1}} - 2 = 1 > 0$$
$$x > \tfrac{9}{4}\ : \quad f'(9) = \frac{3}{\sqrt{9}} - 2 = -1 < 0$$

[5] It is also important to remember that the function is only defined for $x > 0$, so we cannot use 0 or negative numbers as x-values.

Therefore, the sign table looks like this

x :		0		$\frac{9}{4}$	
$f'(x)$:			$+$	0	$-$
$f(x)$:			$\nearrow$	maks.	$\searrow$

The hatched region illustrates that the function is only defined for $x > 0$.

Using the sign table, we see that the graph is increasing until $x = \frac{9}{4}$, and then decreasing. So, the function has a global maximum at $x = \frac{9}{4}$. The y-coordinate of this maximum is

$$f\left(\frac{9}{4}\right) = 6 \cdot \sqrt{\frac{9}{4}} - 2 \cdot \frac{9}{4} = 6 \cdot \frac{3}{2} - \frac{9}{2} = \frac{9}{2} \,.$$

Thus, the function has a global maximum at $\left(\frac{9}{4}, \frac{9}{2}\right)$.

4.2 Inflectional tangents

If we look at the previous examples, it might seem that whenever a graph has a horizontal tangent, it changes from increasing to decreasing or vice versa. However, this is not always this case as demonstrated by the next example.

Example 4.9 Here, we look at the function

$$f(x) = x^3 - 12x^2 + 48x - 62$$

to determine its properties of monotonicity.

First, we determine $f'(x)$:

$$f'(x) = 3x^2 - 12 \cdot 2x + 48 \cdot 1 = 3x^2 - 24x + 48 \,,$$

and then we solve $f'(x) = 0$, which is the quadratic equation

$$3x^2 - 24x + 48 = 0 \,.$$

This equation has only one solution, which is

$$x = 4 \,.$$

Next, we determine the sign of $f'(x)$ for $x < 4$ and for $x > 4$,

$$\begin{aligned} x < 4 :&\quad f'(0) = 3 \cdot 0^2 - 24 \cdot 0 + 48 = 48 > 0 \\ x > 4 :&\quad f'(5) = 3 \cdot 5^2 - 24 \cdot 5 + 48 = 3 > 0 \,. \end{aligned}$$

Then, the sign table looks like this

x :		4	
$f'(x)$:	$+$	0	$+$
$f(x)$:	$\nearrow$	?	$\nearrow$

$f'(4) = 0$, so we have a horizontal tangent at $x = 4$, but we have neither a maximum nor a minimum, because the function is increasing both before $x = 4$ and after $x = 4$. Figure 4.6 illustrates this situation.

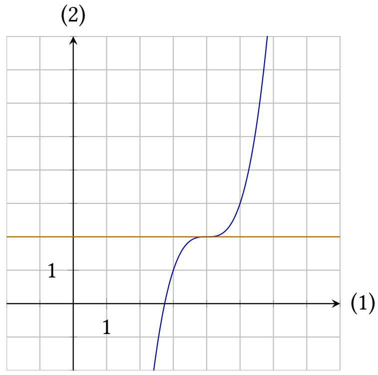


Figure 4.6: At the point $(4, f(4))$ the graph of $f(x) = x^3 - 12x^2 + 48x - 62$ has an inflection point.

In this case, we say that the graph has a *horizontal inflectional tangent*, and the point is called a horizontal inflection point. Therefore, the sign table looks like this,

x :		4	
$f'(x)$:	+	0	+
$f(x)$:	↗	infl.	↗

and the function f is increasing for all values of x.

When we look at figure 4.6, we see clearly that something happens to the graph at the inflection point. The curvature of the graph changes. In the figure, we see that the graph looks like ◜ before the inflection point, and ◞ after the inflection point.

So, a horizontal inflection point is a point where the graph changes its curvature. At this point, the graph also has a horizontal tangent, the so-called inflectional tangent mentioned above.

4.3 Summary of the method

We conclude this section with a general description of how to determine properties of monotonicity and turning points for a given function $f(x)$:

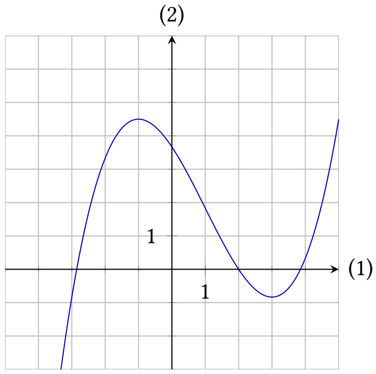
1. Determine $f'(x)$.
2. Solve the equation $f'(x) = 0$. The solutions show where we have possible turning points.[6]
3. The solutions to the equation $f'(x)$ divides the x-axis into a series of intervals. Determine the sign of $f'(x)$ for each of these by inserting some number from the interval into the formula of $f'(x)$.

   We can also choose to draw the graph to investigate how the function behaves in the intervals of monotonicity. In this case, this calculation and the sign table are unnecessary.
4. Draw a sign table.
5. Use the sign table to write a conclusion. If we want to determine a maximum or a minimum, we must remember to also calculate the y-coordinate of the point.

[6]Remember that a solution might also correspond to an inflection point.

4.4 Exercises

Exercise 4.1
The figure below shows the graph of the function $f(x)$.



Find the function's properties of monotonicity.

Exercise 4.2
Determine the properties of monotonicity of the following functions:

a) $f_1(x) = \ln(x) - x + 3\,, \quad x > 0$
b) $f_2(x) = -x^3 - 3x^2 + 9x$
c) $f_3(x) = 5x - e^x\,, \quad -4 \leq x \leq 8$
d) $f_4(x) = -x^3 + 4x^2 + 3x - 3$
e) $f_5(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2 + 3$
f) $f_6(x) = x + \dfrac{16}{x}$
g) $f_7(x) = 4\sqrt{x} - \frac{1}{2}x^2\,, \quad x \geq 0$
h) $f_8(x) = -x^3 + 3x^2 + 4$

Exercise 4.3
Determine the local turning points of the functions in exercise 4.2.

Exercise 4.4
Draw the graph of the function $f(x) = -x^3 + 4x^2 + 3x - 3$, and determine what the number a should be for the equation $f(x) = a$ to have exactly 3 solutions.

Exercise 4.5
Determine the properties of monotonicity and the turning points of the following functions. Pay attention to whether the function is defined for all values of x.

a) $f_1(x) = 3x^2 - 6x + 7.$
b) $f_2(x) = \ln(x) - \frac{1}{2}x^2.$
c) $f_3(x) = x^3 + 3x^2 - 9x + 2$
d) $f_4(x) = \dfrac{1}{2x-4}.$
e) $f_5(x) = 6\sqrt{x} - 2x.$
f) $f_6(x) = x^3 - 12x\,, \quad x \in [-1; 10].$

Exercise 4.6
The function f is given by

$$f(x) = -2x^3 - 4x\,.$$

Use the derivative to argue that f is a decreasing function.

Exercise 4.7
The function g is given by

$$g(x) = x^2\,.$$

Use the derivative to argue that g is not a monotonous function.

Exercise 4.8
Below a sign table is shown for the function f.

x :		-2		1	
$f'(x)$:	$+$	0	$-$	0	$+$

Draw a possible graph for this function.

5 Optimisation

In the last chapter, we described how to find the turning points of a function. We can use this method to *optimise* a given quantity. The purpose of optimisation is to find out when some given quantity is as large or as small as possible.

If the quantity we wish to optimise is given as a function of one variable, all we need to do is determine the maximum or the minimum. However, things are not always this simple. E.g. if we want to determine when a given area is as large as possible, the area might depend on both a length and a width. If this is the case, we need to know how the length and the width are connected.

How we actually do this depends very much on the given situation, and is most easily illustrated by examples.

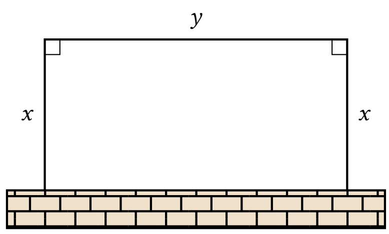


Figure 5.1: A fence around a chicken coop. One side of the area is walled.

Example 5.1 In a garden, we want to build a fence around a chicken coop (see figure 5.1). One side of the garden is walled, so we need only fence 3 sides of a rectangle. If we have 20 m of fence, how should we build the fence, so the enclosure has the largest possible area?

The length and the width of the rectangle, which make up the chicken coop, we call x and y, see figure 5.1. The total length of the fence must then correspond to the length of the three sides, i.e. $2x + y$. Since our fence is 20 m, we have

$$2x + y = 20 \ .$$

Isolating y in this equation yields

$$y = 20 - 2x \ .$$

The area of the rectangle is $A = x \cdot y$, and this is the quantity that needs to be as large as possible. The quantity depends on two variables, x, and y, so we cannot determine the largest value straight away. But we just found out that $y = 20 - x$, therefore this area may also be calculated as

$$A = x \cdot y = x \cdot (20 - 2x) = 20x - 2x^2 \ ,$$

and this expression only depends on x.[1]

Where does this area have a maximum? To find the possible turning points of the function, we employ the method used in the previous chapter, i.e. we solve $A' = 0$.

[1]Notice that $0 < x < 10$. We have $x > 0$ because x is a length, and we have $x < 10$ because we only have 20 m of fence. The two sides with length x must therefore have a total length of less than 20 m. This means that solutions for x which are not in the interval from 0 to 10, must be discarded.

Since $A = 20x - 2x^2$, we get

$$A' = 20 \cdot 1 - 2 \cdot 2x = 20 - 4x ,$$

So, the equation $A' = 0$ is

$$20 - 4x = 0 \qquad \Longleftrightarrow \qquad x = 5 .$$

Now we know that there is a possible maximum for the area at $x = 5$. We graph $A = 20x - 2x^2$ (see figure 5.2), and here we clearly see that $x = 5$ corresponds to a maximum.

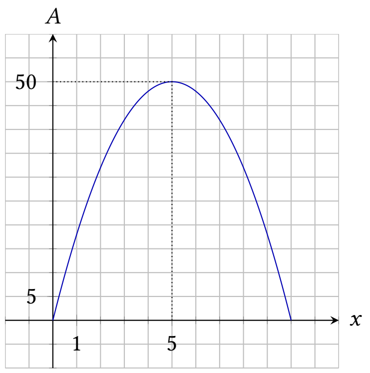


Figure 5.2: At $x = 5$, we have the largest area.

Therefore, the area has a maximum at $x = 5$. This then gives us $y = 10$, and the area $A = 50$, which we also see in the figure.

Example 5.2 We want to build a cylindrical container with a volume of 1 litre and use as little material as possible. We can assume that the used material has the same thickness every—which means that we use the least amount of material, when the surface area is as small as possible.

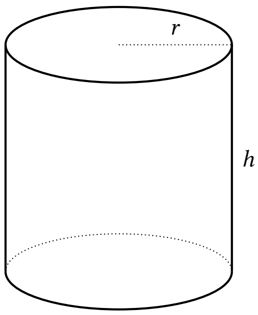


Figure 5.3: A cylinder can be described by its height and radius.

A cylinder can be described by two parameters: Its radius r (at the top and the bottom) and its height h, see figure 5.3. Since the volume is measured in litres, and 1 l = 1 dm^3, r and h are measured in decimetres.

The volume of a cylinder is

$$V = \pi r^2 h ,$$

and since the volume is 1 l, we have

$$\pi r^2 h = 1 \qquad \Longleftrightarrow \qquad h = \frac{1}{\pi r^2} . \tag{5.1}$$

The surface area of a cylinder is

$$A = 2\pi r^2 + 2\pi r h .$$

If we insert the expression for h from (5.1), we get

$$A = 2\pi r^2 + 2\pi r \cdot \frac{1}{\pi r^2} = 2\pi r^2 + \frac{2}{r} .$$

Now, the area is a function of r. Where the area is smallest, we have $A' = 0$. Since

$$A' = 4\pi r - \frac{2}{r^2} ,$$

we therefore have the equation

$$4\pi r - \frac{2}{r^2} = 0 ,$$

which has the solution

$$r = \sqrt[3]{\frac{1}{2\pi}} \approx 0.54 \text{ dm} .$$

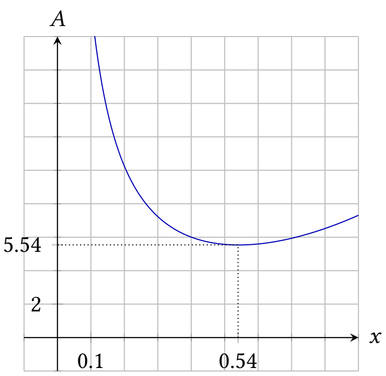


Figure 5.4: We have the smallest surface area, when the radius is 0.54 dm.

That this is indeed a minimum can be seen in figure 5.4.

When we know the radius, $r = 0.54$ dm, we can calculate the height, since equation (5.1) gives us

$$h = \frac{1}{\pi \cdot 0.54^2} = 1.08 \text{ dm} .$$

A cylindrical container with a volume of 1 l, therefore, has the least surface area, when the radius is $r = 0.54$ dm and the height is $h = 1.08$ dm.

Example 5.3 In a garden, we want to plant a 10 m^2 flower bed. The shape of the flower bed is a figure made of a rectangle and a half circle, see figure 5.5.

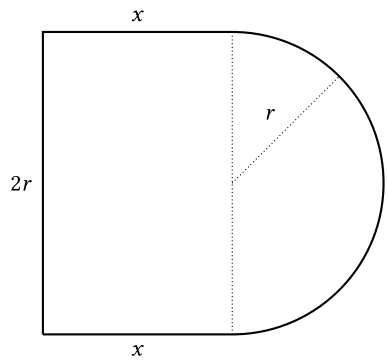


Figure 5.5: A 10 m^2 flower bed is made up of a rectangle and a half circle.

We want to place decorative stones around the edge of the flower bed, so we want to minimise the perimeter. In that case, what are then the lengths of x and r in the figure?

The flower bed is made up of a rectangle with sides x and $2r$, and a half circle, with radius r. Its area is then

$$A = 2r \cdot x + \frac{\pi r^2}{2} .$$

Since the area is 10 m^2, this is equal to 10. Next, we isolate x.

$$2rx + \frac{\pi r^2}{2} = 10 \qquad \Longleftrightarrow \qquad x = \frac{5}{r} - \frac{\pi r}{4} . \tag{5.2}$$

We want to minimise the perimeter. Since the perimeter consists of three straight lines and a half circle, the perimeter is

$$O = 2r + 2x + \pi r .$$

We insert the expression for x from (5.2) into this expression for the perimeter, and we get

$$O = 2r + 2 \cdot \left(\frac{5}{r} - \frac{\pi r}{4}\right) + \pi r = \left(2 + \frac{\pi}{2}\right) r + \frac{10}{r} .$$

To minimise this expression, we differentiate and find

$$O' = 2 + \frac{\pi}{2} - \frac{10}{r^2} .$$

We set this equal to 0, and get the equation

$$2 + \frac{\pi}{2} - \frac{10}{r^2} = 0 ,$$

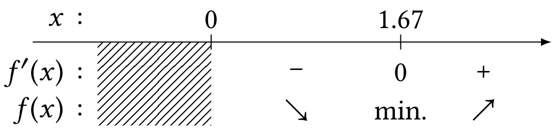
which has the solution

$$r = \frac{10}{\sqrt{20 + 5\pi}} \approx 1.67 \text{ m} .$$

To find out, if this really *is* a minimum, we construct a sign table for O' for values of x greater than or less than 1.67.

$$\begin{aligned} 0 < r < 1.67 : \quad & O'(1) = 2 + \frac{\pi}{2} - \frac{10}{1^2} = -6.43 < 0 \\ r > 1.67 \quad & O'(2) = 2 + \frac{\pi}{2} - \frac{10}{2^2} = 1.07 > 0 \end{aligned}$$

The sign table looks like this



and we have a minimum, when the radius of the circle $r = 1.67$ m.

The length x is then (we use the result from (5.2))

$$x = \frac{5}{1.67} - \frac{\pi \cdot 1.67}{4} = 1.67 \text{ m} .$$

5.1 Summary of the method

The method we used in the examples above can be described in the following way.

1. Translate a condition (e.g. fixed perimeter, fixed area, fixed volume) into an equation. Then isolate one of the variables in this equation.
2. Write down an expression for the quantity you wish to optimise, and replace one of the variables with the expression found in step 1. You now have a function of one variable.
3. Determine the turning points of the function found in step 2. Now, you can determine the remaining measurements.

In principle, it is possible to have more than two variables in the expression, we wish to optimise. Then we need more than one condition to write the expression as a function of one variable. This corresponds to repeating steps 1–2.

5.2 Exercises

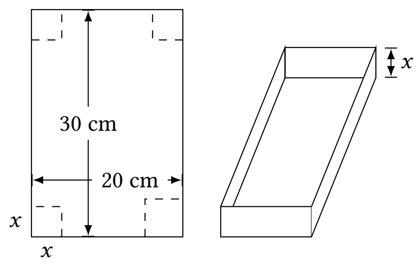
Exercise 5.1
If $x + y = 64$, what is the largest possible value of $x \cdot y$?

Exercise 5.2
A rectangular area is to be fenced off by 400 m of fence. The area has to be as large as possible. Determine the length and the width of the area.

Exercise 5.3
A box has a volume of 1200 cm^3. The box has a square bottom and no lid.

Determine the volume of the box if it is built using the least amount of materials.

Exercise 5.4
A piece of cardboard is folded into a box by cutting away a square at each corner (see figure).



Determine the value of x, so that the volume of the box is as large as possible.

Exercise 5.5
Two numbers x and y satisfy the equation $x + y = 10$.

a) What must x and y be for $x^2 + y^2$ to be as large as possible?
b) What must x and y be for $x^2 + y^2$ to be as small as possible?

Exercise 5.6
We want to build a box where the sides at the bottom are in a ratio of 1:3. The material for the top and the bottom costs £6 per m^2, and the material for the sides costs £8 per m^2.

Determine the dimensions of the box if it costs £50 and has the largest possible volume.

Exercise 5.7
A soda can is shaped like a cylinder. The can contains 330 ml (1 ml = 1 cm^3).

a) What would the dimensions of the can be if we just want to minimise the surface area?

In reality, a soda can is not a perfect cylinder. Some extra material is used to make the opening at the top, and the bottom is not flat, but curved. We can simulate this by assuming that the top and the bottom have a different thickness compared to the side. (A real soda can is actually about 11.5 cm tall with a diameter of approximately 6.4 cm.)

b) What would the dimensions be if the top and the bottom are twice as thick as the side, and we want to use the least amount of material?
c) What else might be taken into consideration, when a soda can is designed?

Exercise 5.8
A rectangular warehouse is to be built with a floor area of 5000 square metres. The warehouse will be divided into two rectangular rooms by means of an inner wall.

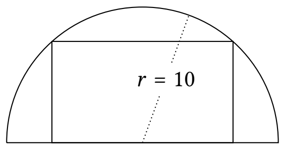
It costs £600 per metre to build the outer walls and £350 per metre to build the inner wall.

Find the dimensions of the warehouse, so that the cost of building it is as low as possible.

Exercise 5.9
A 50 cm string is cut into two pieces. One of the pieces is shaped like a circle and the other as a square.

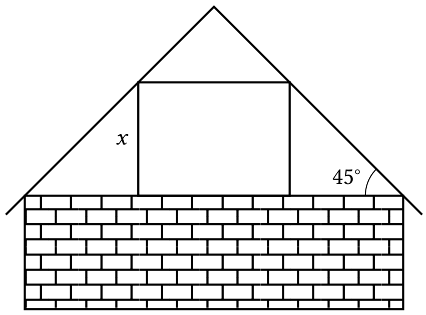
a) Where should we cut the string if the combined area of the two figures is to be as large as possible?
b) Where should we cut the string if the combined area of the two figures is to be as small as possible?

Exercise 5.10
A rectangle is inscribed in a half circle with radius 10 (see figure).



Determine the dimensions of the rectangle if its area is to be as large as possible.

Exercise 5.11
A house is 10 m wide and its roof pitch is 45° (see figure). On the first floor, a rectangular glass section is to be built.



a) What is the height x if the glass section is to be as large as possible?
b) What is x if the roof pitch is 50°?

Exercise 5.12
A company sets up a model for their production of x units of a specific product. In the model, $O(x)$ denotes the combined cost (in DKK) of production, and $p(x)$ denotes the price (DKK per unit) the product must have for all units to be sold.

It turns out that

$$O(x) = 0,0025 \cdot x^2 + 10^6$$

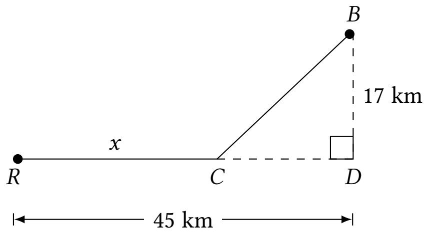
and

$$p(x) = -0,007x + 1400 ,$$

where x is the number of units sold.

a) Find an expression for the total turnover $A(x)$.
b) Find an expression for the profit $F(x)$, i.e. the turnover minus the cost.
c) How many units does the company need to produce for the profit to be as large as possible?

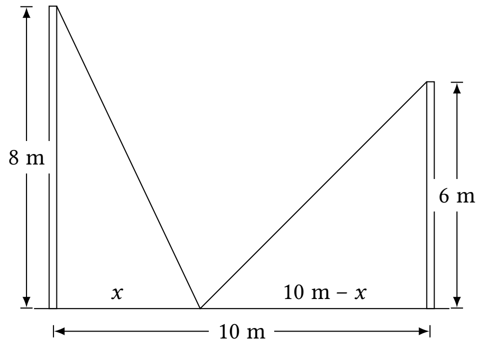
Exercise 5.13
An oil rig (B) is placed 17 km from the coast. A pipe line must be built from the oil rig to a refinery (R), which lies 45 km away along the coast (see figure).



It is more expensive to build a pipeline under water than on land. The question is now where along the coast (at point C) the pipeline should hit the shore for the total cost to be minimised.

a) If it is twice as expensive to build a pipeline under water than on land, what is then the value of x?

Exercise 5.14
Two poles are placed 10 m apart. The poles are stabilised by a wire attached to the ground at a point between the two poles. Some of the measurements are shown in the figure below.



Determine the value of x which minimises the length of the wire.

6 Rate of change

Using differentiation, we may find out where certain quantities have maxima and minima. This can, for instance, be used for optimisation. But we can also use differentiation to determine how fast certain quantities grow at certain points.

We have the following definition.[1]

[1]Note that in this definition, the independent variable is called t instead of x. In principle, we could have used x, but we use t to emphasise that we are talking about *time*.

> **Definition 6.1**
>
> Let $f(t)$ be a function, where t is the time. Then $f'(t)$ is the *rate of change* at the time t.

Example 6.2 In figure 6.1, we see the graph of $f(t)$, which shows us how the amount of sparrows on a certain island increases over time (measured in years).

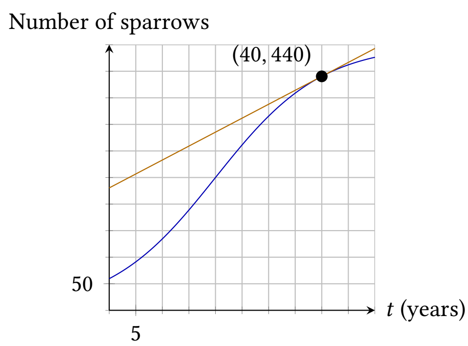


Figure 6.1: The population of sparrows at time t (in years).

In the figure, we see the graph passing through the point $(4, 440)$. We have also drawn a tangent through this point—the slope of the tangent is 5.25. In other words

$$f(40) = 440 \qquad \text{and} \qquad f'(40) = 5.25 \,.$$

This is a purely mathematical description, which may be translated into

1. After 40 years, there are 440 sparrows on the island.
2. After 40 years, the amount of sparrows increases at a rate of 5.25 sparrows per year.

Example 6.3 A jug of lukewarm water is put into a refrigerator. The temperature of the water can then be described by the function

$$f(t) = 5 + 15 \cdot e^{-0.01 \cdot t} \,,$$

where the time t is measured in minutes.

From this function, we can determine the rate of change $f'(45)$. First we calculate

$$f'(t) = 0 + 15 \cdot (-0.01) \cdot \mathrm{e}^{-0{,}01 \cdot t} = -0.15 \cdot \mathrm{e}^{-0.01 \cdot t} \,,$$

and then

$$f'(45) = -0.15 \cdot \mathrm{e}^{-0.01 \cdot 45} = -0.096 \,.$$

What does this number tell us?

First of all, we notice that the number is negative, i.e. the temperature is *decreasing*. The value of the number shows us how much. Since $f'(45) = -0.096$, we have the following interpretation:

After 45 minutes in the refrigerator, the temperature of the water decreases at a rate of 0.096°C per minute.

6.1 Exercises

Exercise 6.1
A new year's rocket is fired vertically upwards. The height h of the rocket (measured in metres) as a function of the time t (measured in seconds) may be described by the model

$$h(t) = -4.9 \cdot t^2 + 45t \ , \quad 0 \le t \le 4.5 \ .$$

a) What is the speed of the rocket after 1 s?

b) What is the speed of the rocket after 2 s?

c) What is the speed of the rocket when it has reached a height of 75 m?

Exercise 6.2
A patient is given an injection of a certain type of medicine. At the time t (measured in hours after injection), the concentration c (in ng/L) of the medicine in the bloodstream can be described by the model

$$c(t) = 120 \cdot 0.87^t \ .$$

a) Determine the number $c'(2)$, and give an interpretation of this number.

b) At what time does the medicine concentration decrease at a rate of 3 ng/L per second?

More derivatives

A

In this section, we show how to find the derivatives of $\ln(x)$, e^x, a^x and x^n. To find the first of these, we use the three-step method—the rest are found using the rules in sections 2.4 and 2.5.

This will prove the last of the claims in table 2.6.

Theorem A.1

If $f(x) = \ln(x)$, the derivative is $f'(x) = \frac{1}{x}$.

Proof

Here, we use the three-step method. First, we find[1]

$$\Delta f = \ln(x + \Delta x) - \ln(x) = \ln\left(\frac{x + \Delta x}{x}\right) = \ln\left(1 + \frac{\Delta x}{x}\right) .$$

[1] In this calculation, we use the rule

$$\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right) .$$

Next, we look at

$$\frac{\Delta f}{\Delta x} = \frac{\ln\left(1 + \frac{\Delta x}{x}\right)}{\Delta x} = \frac{1}{\Delta x} \cdot \ln\left(1 + \frac{\Delta x}{x}\right) . \qquad (A.1)$$

We cannot simplify this further.

Now, we need to let $\Delta x \longrightarrow 0$, but the expression (A.1) is too complicated to see what that gives us. We therefore use a little trick: We introduce a new variable t, which is equal to $\frac{\Delta x}{x}$. Letting $\Delta x \longrightarrow 0$ corresponds to letting $t \longrightarrow 0$.

(A.1) can now be rewritten as

$$\frac{\Delta f}{\Delta x} = \frac{1}{xt} \cdot \ln(1 + t) ,$$

which then corresponds to

$$\frac{\Delta f}{\Delta x} = \frac{1}{x} \cdot \frac{1}{t} \cdot \ln(1 + t) = \frac{1}{x} \cdot \ln\left((1 + t)^{\frac{1}{t}}\right) . \qquad (A.2)$$

It is well-known that[2]

$$(1 + t)^{\frac{1}{t}} \longrightarrow e \quad \text{when} \quad t \longrightarrow 0 . \qquad (A.3)$$

Actually, this is sometimes used as the definition of the number e. We are not going to prove the result in (A.3), but that it is correct may be inferred from the graph of $(1 + t)^{\frac{1}{t}}$ shown in figure A.1.

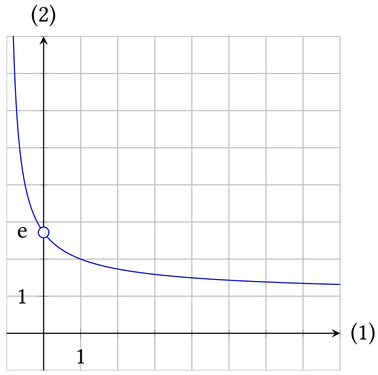


Figure A.1: The graph of $(1 + t)^{\frac{1}{t}}$.

Now, letting $\Delta x \longrightarrow 0$ is the same as letting $t \longrightarrow 0$ in (A.2), and using the result from (A.3), we get

$$f'(x) = \frac{1}{x} \cdot \ln(e) = \frac{1}{x} \, . \qquad \blacksquare$$

Theorem A.2

When $f(x) = \mathrm{e}^x$, the derivative is $f'(x) = \mathrm{e}^x$.

Proof
Since e^x is the inverse of $\ln(x)$, we have the following equation:

$$\ln(\mathrm{e}^x) = x \, . \tag{A.4}$$

If we differentiate both sides of this equation, it will still hold.

On the left hand side, we need to differentiate a composite function, and we get[2]

$$(\ln(\mathrm{e}^x))' = \frac{1}{\mathrm{e}^x} \cdot (\mathrm{e}^x)' = \frac{1}{\mathrm{e}^x} \cdot f'(x) \, .$$

[2]Here, it is important to remember that we do not yet know the derivative of e^x. Therefore, we must write $(\mathrm{e}^x)'$, which is the same as $f'(x)$.

On the right hand side we get

$$(x)' = 1 \, .$$

Since the left hand side is equal to the right hand side, we have

$$\frac{1}{\mathrm{e}^x} \cdot f'(x) = 1 \qquad \Longleftrightarrow \qquad f'(x) = \mathrm{e}^x \, . \qquad \blacksquare$$

Theorem A.3

If $f(x) = a^x$, where $a > 0$, then $f'(x) = \ln(a) \cdot a^x$.

Proof
We can rewrite the function f as

$$f(x) = a^x = \left(\mathrm{e}^{\ln(a)}\right)^x = \mathrm{e}^{\ln(a) \cdot x} \, .$$

This is a composite function, and its derivative is

$$f'(x) = \mathrm{e}^{\ln(a) \cdot x} \cdot \ln(a) = a^x \cdot \ln(a) = \ln(a) \cdot a^x \, . \qquad \blacksquare$$

Theorem A.4

If $f(x) = x^n$, the derivative is $f'(x) = nx^{n-1}$.

Proof
First, we rewrite the formula for $f(x)$:

$$f(x) = x^n = \mathrm{e}^{\ln(x^n)} = \mathrm{e}^{n \cdot \ln(x)} \, .$$

So, f can be written as a composite function, where the outer function is

$$p(q) = \mathrm{e}^q ,$$

and the inner function is

$$q(x) = n \cdot \ln(x) ,$$

where n is a constant.

If we differentiate the outer function, we get

$$p'(q) = \mathrm{e}^q .$$

The inner function yields

$$q'(x) = n \cdot \frac{1}{x} .$$

So,

$$\begin{aligned} f'(x) = p'(q) \cdot q'(x) &= \mathrm{e}^q \cdot n \cdot \frac{1}{x} \\ &= \mathrm{e}^{n \cdot \ln(x)} \cdot n \cdot \frac{1}{x} = x^n \cdot n \cdot \frac{1}{x} \\ &= n \cdot x^{n-1} . \end{aligned}$$

■

Bibliography

[1] Robert A. Adams. *Calculus: a complete course.* 3rd ed. Addison-Wesley Publishers Limited, 1995.

[2] Demetrios Kanoussis. *Euler's Number e.* May 9, 2015. URL: http://www.goldenratiopublications.com/eulers-number-e/ (visited on 08/03/2015).