Arithmetic

Version 1.1

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$$\frac{7}{2} + \frac{3}{4} \cdot 2$$

$$4 + 6^2 - 3 \cdot (-7)$$

$$\frac{4x^2y}{2xy^5}$$

$$2x + 4 = 3$$

$$(x + 3)(2x - 1) = 0$$

$$\sqrt[3]{7 + 2 \cdot 5}$$

$$(a + b)(a - b)$$

$$a^2 + b^2 - 2ab$$

Mike Vandal Auerbach

www.mathematicus.dk

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Version 1.1, 2019

These notes are a translation of the Danish "Regning" written for the Danish stx.

The notes cover basic arithmetical and algebraic techniques which are necessary for the understanding of more advanced topics.

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The document is written using the document preparation system LaTeX, see www.tug.org and www.miktex.org. Figures and diagrams are produced using *pgf/TikZ*, see www.ctan.org/pkg/pgf.

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Contents

1 Numbers and Arithmetic

Numbers are one of the cornerstones of mathematics. Numbers are used for many different purposes: For counting and measuring or to calculate profit and debt.

In the next few sections, we look at different types of numbers and the four arithmetical operations: *addition*, *subtraction*, *multiplication* and *division*.

The first numbers we learn are the numbers we call *natural numbers*. These are the numbers used for counting, i.e.

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \dots \ .$$

If we want to express the notion of nothing, we use the number 0. However, we do not place this among the natural numbers.

1.1 Addition

The simplest of the four arithmetical operations is addition. This operation expresses the same idea as the word "and".[1]

[1]The symbol + in all likelihood comes from the word *et* which means "and" in latin.[1]

E.g. if we have a pile of 7 items and a pile of 11 items, we have a total of "7 items and 11 items", i.e. 18 items. In mathematics we write

$$7 + 11 = 18 \ .$$

The two numbers 7 and 11 are called *terms* and the result 18 is called a *sum*.

Since the sign + tells us to add the values, the order of the terms cannot matter. Therefore, we expect that

$$7 + 11 = 11 + 7 \ ,$$

which is true.

1.2 Subtraction and Negative Numbers

Using the natural numbers, we cannot express the idea of loss or debt. To do this, we need the negative numbers.[2] We begin by looking at negative whole numbers:

[2]The negative numbers are actually quite a new invention in mathematics. Well into the 18th century, some mathematicians believed that negative numbers did not actually exist.[2]

$$-1, -2, -3, -4, -5, -6, \dots \ .$$

In a way, every negative number is an "opposite number" of a positive number. In mathematics we call this an *inverse* number (with respect to addition). Adding a number and its inverse yields e.g. [3]

[3]In the calculation, we write parentheses around the number −3. We do this to show that − is a sign, i.e. it belongs to the number −3. Generally, we may not write a sign without parentheses directly after an arithmetical operation.

$$3 + (-3) = 0 \quad \text{and} \quad (-3) + 3 = 0 \ .$$

We may argue that −(−3) must be the inverse of −3 and that we therefore must have

$$-(-3) + (-3) = 0 \ .$$

But because 3 + (−3) = 0, we must also have

$$-(-3) = 3 \ .$$

This applies to all numbers (not just the number 3).

Using the negative numbers, we may *define* subtraction as adding the inverse number. An example might be

$$8 - 2 = 8 + (-2) \ .$$

This also explains why the numbers are not directly interchangeable. 2 − 8 is not the same as 8 − 2 because the sign − actually belongs to the number 2. When we add, however, the order does not matter. Thus we can do as follows:

$$8 - 2 = 8 + (-2) = -2 + 8 \ .$$

Here we see that the sign − is always before the number 2.

The two numbers 8 and 2 in the calculation are called *terms* (as with addition) while the result (6) is called a *difference*.

1.3 Multiplication

Multiplication may be viewed as an extension of addition since e.g.

$$7 \cdot 4 = \overbrace{4 + 4 + 4 + 4 + 4 + 4 + 4}^{7 \text{ times}} = 28 \ .$$

Here, we see why the symbol "·" is called "times".

For *addition*, we may easily argue that the order does not matter. This is also the case for multiplication, though it may not be as obvious why e.g. 7 · 4 = 4 · 7. However, if we view the number 4 · 7 as the area of a rectangle where one side is 4 and the other 7, exchanging the numbers 4 and 7 only amounts to rotating the rectangle—hence the area stays the same. This is illustrated in figure 1.1.

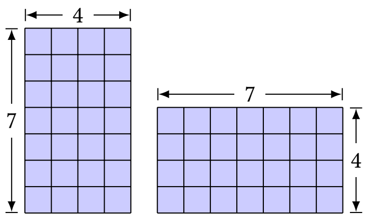


Figure 1.1: From this figure, we may argue that 7 · 4 = 4 · 7.

The numbers that are multiplied (7 and 4) are called *factors* and the result (28) is called the *product* of the two numbers.

Sign

It is easy to show what happens when we multiply positive numbers. But what happens if there are negative numbers involved?

Negative numbers may be interpreted as debt. In this light, a calculation like $-6 \cdot 3$ should be interpreted as a debt of 6 items multiplied by 3, i.e. a debt of 18 items. Thus we have

$$-6 \cdot 3 = -18 \; .$$

Since the order of multiplication does not matter, we must also have[4]

$$3 \cdot (-6) = -18 \; .$$

[4]Notice the parentheses around −6. We write this to indicate that the sign belongs to the number 6 (and it is not optional).

From this, we see that multiplying a positive by a negative number yields a negative number.

But what happens when we multiply two negative numbers? Here, we can use the following argument: $-2 \cdot (-4)$ is the (additive) inverse of $2 \cdot (-4)$. And because $2 \cdot (-4) = -8$, we have

$$-2 \cdot (-4) = -(-8) = 8 \; .$$

Hence, when we multiply two negative numbers, the result is positive. The sign rules for multiplication are listed in table 1.2.

Table 1.2: Sign rules for multiplication.

Rule	Example
$(+) \cdot (+) = (+)$	$2 \cdot 3 = 6$
$(+) \cdot (-) = (-)$	$2 \cdot (-4) = -8$
$(-) \cdot (+) = (-)$	$(-3) \cdot 5 = -15$
$(-) \cdot (-) = (+)$	$(-4) \cdot (-2) = 8$

1.4 Division

If 2 people share 6 items, they get 3 each. The calculation we perform is a division:

$$\frac{6}{2} = 3.$$

The number which we divide (6) is called the *dividend*, and the number we divide by (2) is called the *divisor*. The result of a division is called a *quotient*.

Division is the opposite of multiplication. The number we find in the calculation above is the answer to the question: *Which number do we multiply by 2 to get 6?*[5]

[5]This explains why we cannot divide by 0. The result of the calculation $\frac{4}{0}$ answers the question: *What do we multiply by 0 to get 4?* But a number multiplied by 0 is always 0, i.e. a number multiplied by 0 can never be 4. Therefore it makes no sense to divide by 0.

The result of a division is not necessarily a whole number. We therefore have to introduce some other numbers, namely fractions, i.e. numbers like $\frac{1}{2}$, $\frac{5}{3}$ and $-\frac{7}{13}$.

Fractions are not as easy to work with as whole numbers; how do we add, e.g., $\frac{1}{2}$ and $\frac{2}{3}$? How we calculate with fractions can be deduced from what is written above. However, there are quite a few things to deduce, so this will be explained in a later section.

Sign

Division may be viewed as a form of multiplication. The calculation $\frac{6}{2} = 3$ may also be written as

$$6 \cdot \frac{1}{2} = 3 \; .$$

But if every division can be turned into a multiplication, the same sign rules must apply.

Therefore e.g.

$$\frac{-20}{5} = -4, \qquad \frac{14}{-2} = -7 \qquad \text{and} \qquad \frac{-18}{-6} = 3\ .$$

The sign rules for division are listed in table 1.3.

Table 1.3: Sign rules for division.

Rule	Example
$\frac{(+)}{(+)} = (+)$	$\frac{6}{2} = 3$
$\frac{(+)}{(-)} = (-)$	$\frac{10}{-5} = -2$
$\frac{(-)}{(+)} = (-)$	$\frac{-14}{2} = -7$
$\frac{(-)}{(-)} = (+)$	$\frac{-18}{-3} = 6$

Decimal Numbers

Non-integers are often written as decimal numbers instead of fractions. An example of a decimal is the number 1.472.

In principle, a decimal number is a sum of fractions, e.g.

$$1.472 = 1 + \frac{4}{10} + \frac{7}{100} + \frac{2}{1000}\ .$$

However, this is not something we need to think about when we use decimals.

Actually, every number may be written as a decimal number. We have

$$\frac{1}{2} = 0.5$$
$$\frac{1}{3} = 0.333333333\ldots$$
$$\frac{10}{7} = 1.42857\,142857\,142857\ldots$$

As we see from the last two numbers, we sometimes need an infinite amount of decimals to write a number as a decimal number. For this reason, a fraction is more precise than a decimal number.[6]

[6]This also applies to $\frac{1}{2}$. Even though this can be written as 0.5, we are more precise when we write $\frac{1}{2}$. If we write 0.5, it is impossible for a reader to see if the number actually has an infinite amount of zeroes after the 5, or if it is the result of a rounding from e.g. 0.496.

What we also see from the decimal numbers above is that even though e.g. $\frac{10}{7}$ cannot be written precisely as a decimal number, there is a recurring pattern in the decimals. We write the same sequence of numbers over and over. This is true for every fraction when we write it as a decimal number; if a fraction is written as a decimal number it either has a finite amount of decimals or the decimals repeat the same pattern ad infinitum. We call these numbers *rational numbers*, i.e. numbers that can be written as a *ratio* of two whole numbers.

Irrational Numbers

If we write a fraction as a decimal, we either get a finite amount of decimals or an endlessly repeating pattern. From this follows that a decimal number without any pattern in the decimals cannot be written as a fraction.

But is it even possible to conceive a number, which cannot be written as a fraction? It turns out that there are infinitely many of such numbers, a well-known example is the number π—the ratio of the circumference and the diameter of a circle. To 20 decimal places, we have

$$\pi = 3,14159265358979323846\ldots\ .$$

Here there is no pattern in the decimals, and there never will be, no matter how many decimals we calculate.

These numbers, which cannot be written as fractions, are called *irrational numbers.* The rational and the irrational numbers collectively make up the *real numbers.* If we view numbers as points on a number line, every point on the line corresponds to a real number.

We can now group numbers into different sets:

The natural numbers which we use for counting: $1, 2, 3, 4, \ldots$.

The whole numbers, including the negatives: $\ldots, -3, -2, -1, 0, 1, 2, \ldots$.

The rational numbers which are numbers that may be written as fractions.[7]

The real numbers, i.e. *all* numbers.

[7]The whole numbers are also rational because every whole number can be written as a fraction; e.g. $4 = \frac{8}{2}$ and $-5 = \frac{-15}{3}$.

1.5 Powers and Roots

$4+4+4$ can be written as $4 \cdot 3$. Similarly, a shorthand exists for e.g. $5 \cdot 5 \cdot 5 \cdot 5$. Since we multiply four 5's, we instead write 5^4. I.e.[8]

$$5^4 = \overbrace{5 \cdot 5 \cdot 5 \cdot 5}^{4\text{ times}}.$$

[8]In 5^4, the number 5 is called the *base* and the number 4 is called the *exponent.*

5^4 is called "5 raised to the power 4" or "the 4th power of 5".

The opposite calculation is called a *root* of a number. E.g. we may calculateNotice that it is not called the "2nd root" but the *square root* and we do not write the number 2. I.e. we write $\sqrt{49}$ and not $\sqrt[2]{49}$.

$$\begin{aligned} &\sqrt[4]{81} && \text{the 4th root of 81,} \\ &\sqrt[3]{125} && \text{the cube root of 125,} \\ &\sqrt[5]{32} && \text{the 5th root of 32,} \\ &\sqrt{49} && \text{the square root of 49.} \end{aligned}$$

The results of these calculations are

$$\begin{aligned} \sqrt[4]{81} &= 3 && \text{because } 3^4 = 81 \\ \sqrt[3]{125} &= 5 && \text{because } 5^3 = 125 \\ \sqrt[5]{32} &= 2 && \text{because } 2^5 = 32 \\ \sqrt{49} &= 7 && \text{because } 7^2 = 49\,. \end{aligned}$$

There are a lot of rules we can use when we do calculations with powers and roots; these are described in a later chapter.

1.6 Order of Operations

When we calculate e.g. $7 + 5 \cdot 3^2$, we need to know in which order to do the different steps. Is the first step to add 7 and 5 or to calculate 3^2?

Therefore we have rules that describe the order in which we calculate. This ensures that everybody gets the same (correct) result from a certain calculation.[9]

[9]The order of operations is the one that makes sense logically. We multiply before we add because $4 \cdot 3 = 4 + 4 + 4$. Therefore

$$2 + 4 \cdot 3 = 2 + 4 + 4 + 4\,,$$

and we therefore have to multiply first (unless we want to rewrite every multiplication as an addition).

Theorem 1.1: The order of operations

When we do a calculation, the arithmetical operations are performed in the following order:

1. First we calculate powers and roots,
2. then we multiply and divide,
3. and lastly we add and subtract.

This order can only be changed using parentheses. If some part of a calculation is written in a parenthesis, we view this as a separate calculation, which must be done *first*.

Some examples of calculations are:

Example 1.2 Using the order of operations:

$$\begin{aligned} 2 \cdot 17 - 4 \cdot 2^3 &= 2 \cdot 17 - 4 \cdot 8 && \text{First we calculate } 2^3, \\ &= 34 - 32 && \text{then we multiply,} \\ &= 2 && \text{and subtract.} \end{aligned}$$

Example 1.3 An example including parentheses:

$$\begin{aligned} (6+2) \cdot 5 + 3 \cdot \sqrt{16} &= 8 \cdot 5 + 3 \cdot \sqrt{16} && \text{The parenthesis is calculated first,} \\ &= 8 \cdot 5 + 3 \cdot 4 && \text{then the roots;} \\ &= 40 + 12 && \text{then we multiply,} \\ &= 52 && \text{and lastly, we add.} \end{aligned}$$

Hidden Parentheses

Some calculations actually include parentheses, which are not obviously there.

In a calculation such as $\frac{3+17}{10}$, we have to calculate $3 + 17$ first—before we divide. When we write a division, there are actually parentheses around both the numerator and the denominator, and this must be taken into account when we calculate.

The same applies to roots. In e.g. $\sqrt{17-8}$, we must calculate $17 - 8$ before we take the square root.

Example 1.4 In the following examples, it is shown explicitly where the hidden parentheses are:

$$\begin{aligned} \frac{3+9}{4} &= \frac{(3+9)}{4} \\ \frac{50}{7-2} &= \frac{50}{(7-2)} \\ 7^{2+1} &= 7^{(2+1)} \end{aligned}$$

$$\sqrt{7+9} = \sqrt{(7+9)}$$
$$\sqrt[5-2]{8} = \sqrt[(5-2)]{8}\ .$$

These are important to remember.

The example above does not cover every possible case. But as a rule, if something looks like a separate part of a calculation, it probably is.

1.7 Exercises

Exercise 1.1
Calculate the following:

a) $12 - 5$
b) $2 - 6$
c) $15 - 6 - 7$
d) $7 - (-8)$

Exercise 1.2
Calculate the following:

a) $5 \cdot (-3)$
b) $7 \cdot 2$
c) $-8 \cdot (-4)$
d) $\frac{-12}{4}$
e) $\frac{22}{-11}$
f) $\frac{-18}{-3}$

Exercise 1.3
Calculate the following:

a) 3^4
b) 6^2
c) $\sqrt[3]{27}$
d) $\sqrt[4]{16}$

Exercise 1.4
Calculate the following:

a) $3 - 4 \cdot 2$
b) $5 \cdot 3^2$
c) $11 - 2 \cdot (8+3) - \frac{16}{8} + 1$
d) $9 \cdot (-2) + 10 \cdot 3 - \frac{6}{2} \cdot \left(3 \cdot 2 - 5 \cdot \frac{14}{7}\right)$

2 Fractions

A fraction is a number of equal parts of a whole. We write fractions as two integers above and below a straight line:

$$\frac{2}{3}\,,\quad \frac{7}{4}\,,\quad \frac{-13}{29}\,.$$

The number above is called the *numerator*, and the number below is called the *denominator*.[1]

[1] The numerator and the denominator are always integers, possibly negative. However, the denominator cannot be 0.

The fraction $\frac{2}{3}$ is the number we get when we divide 1 into 3 parts and take 2 of them. A fraction may also be interpreted as the exact result of dividing the numerator by the denominator.

If we want to visualise a fraction, we may do so by using the number line (see figure 2.1).

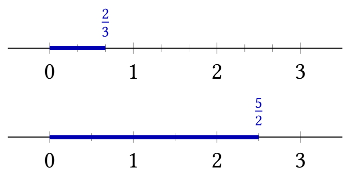


Figure 2.1: The two numbers $\frac{2}{3}$ and $\frac{5}{2}$ on the number line.

2.1 Equivalent Fractions

If we interpret a fraction as the result of dividing the numerator by the denominator, a fraction may actually be equal to a whole number. This is true if the denominator divides the numerator,[2]

[2] Alternatively, whole numbers may be seen as fractions with denominator 1, e.g. $8 = \frac{8}{1}$.

$$\frac{10}{5} = 2\,,\qquad \frac{36}{9} = 4\,,\qquad \frac{-27}{3} = -9\,.$$

Even if this is not the case, we can sometimes write the fraction with a smaller numerator and denominator (*simplifying* the fraction). This is possible if a whole number exists, which divides both the numerator and the denominator.

This situation is illustrated in figure 2.2. Here we see that $\frac{4}{6} = \frac{2}{3}$. In the fraction $\frac{4}{6}$, the number 2 divides the numerator and the denominator. Since a fraction is, in a way, a ratio between the numerator and the denominator, its size does not change if we divide the numerator and the denominator by the same number. We therefore have

$$\frac{4}{6} = \frac{4/2}{6/2} = \frac{2}{3}\,.$$

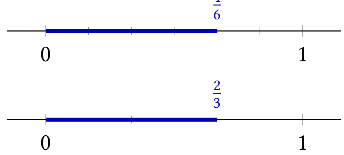


Figure 2.2: From the two number lines, we see that $\frac{4}{6} = \frac{2}{3}$.

This does not change the value of the fraction. The number is exactly the same as before, we just write it using smaller numbers for the numerator and the denominator—and it is always easier to work with smaller numbers.

Example 2.1 A few examples of this kind of reduction are:

$$\frac{15}{36} = \frac{15/3}{36/3} = \frac{5}{12}\,,$$
$$\frac{24}{56} = \frac{24/8}{56/8} = \frac{3}{7}\,,$$
$$\frac{27}{18} = \frac{27/9}{18/9} = \frac{3}{2}\,.$$

If dividing by the same number in the numerator and the denominator does not change the value of the fraction, we may also multiply.[3] If we do this, the numerator and the denominator become larger, and this does not seem very useful. But it turns out to be very useful when we want to add fractions—see the section below.

[3]It is important to remember that we have to divide or multiply by the *same number* in the numerator and the denominator. Otherwise we change the value of the fraction.

Example 2.2 If we multiply $\frac{3}{4}$ by 5 in the numerator and the denominator, we get

$$\frac{3}{4} = \frac{3 \cdot 5}{4 \cdot 5} = \frac{15}{20}\,.$$

Multiplying $\frac{8}{5}$ by $\frac{3}{3}$ yields

$$\frac{8}{5} = \frac{8 \cdot 3}{5 \cdot 3} = \frac{24}{15}\,.$$

2.2 Addition and Subtraction

It turns out that we can only add fractions if they have the same denominator. If this is the case, we just add the numerators. E.g.

$$\frac{2}{5} + \frac{4}{5} = \frac{2+4}{5} = \frac{6}{5}\,.$$

This is illustrated in figure 2.3.

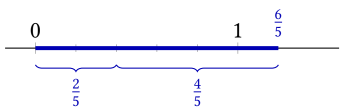


Figure 2.3: The calculation $\frac{2}{5} + \frac{4}{5} = \frac{6}{5}$ illustrated on the number line.

It is not possible to add fractions with different denominators. Nonetheless, we would like to have method for adding e.g. $\frac{1}{4}$ and $\frac{2}{3}$. If we can only add fractions when they have the same denominator, we need a method for producing equal denominators.

We do this by using the method described in the section above. We multiply the fraction $\frac{1}{4}$ by 3 in the numerator and the denominator and multiply $\frac{2}{3}$ by 4. We then get

$$\frac{1}{4} + \frac{2}{3} = \frac{1 \cdot 3}{4 \cdot 3} + \frac{2 \cdot 4}{3 \cdot 4} = \frac{3}{12} + \frac{8}{12} = \frac{11}{12}\,.$$

Here both fractions end up with the numerator 12; they can then be added.

In this calculation we multiply the first fraction with the denominator of the second, and vice versa. This always works.[4]

[4]But it is not always necessary. Sometimes smaller numbers exist which also yield a common denominator.

Example 2.3 A few examples of additions involving fractions:

$$\frac{1}{3} + \frac{1}{2} = \frac{1 \cdot 2}{3 \cdot 2} + \frac{1 \cdot 3}{2 \cdot 3} = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}\,,$$

$$\frac{7}{4}+\frac{2}{11}=\frac{7\cdot 11}{4\cdot 11}+\frac{2\cdot 4}{11\cdot 4}=\frac{77}{44}+\frac{8}{44}=\frac{85}{44}\,,$$
$$\frac{3}{5}+\frac{1}{10}=\frac{3\cdot 2}{5\cdot 2}+\frac{1}{10}=\frac{6}{10}+\frac{1}{10}=\frac{7}{10}\,.$$

In the last calculation, we see that the common denominator is not the product of the two original denominators, but instead the smallest number, which both denominators divide.[5]

[5]Usually, we try to calculate the result using the smallest numbers possible—simply because it is easier.

If we want to subtract fractions, we can use the exact same method. We can only subtract fractions if they have a common denominator. E.g.

$$\frac{8}{13}-\frac{3}{13}=\frac{8-3}{13}=\frac{5}{13}\,.$$

If the two fractions do not have a common denominator, we multiply the numerator and denominator of each fraction to give them a common denominator.

Example 2.4 Three examples of subtracting fractions:

$$\frac{4}{3}-\frac{2}{5}=\frac{4\cdot 5}{3\cdot 5}-\frac{2\cdot 3}{5\cdot 3}=\frac{20}{15}-\frac{6}{15}=\frac{14}{15}\,,$$
$$\frac{7}{8}-\frac{1}{2}=\frac{7}{8}-\frac{1\cdot 4}{2\cdot 4}=\frac{7}{8}-\frac{4}{8}=\frac{3}{8}\,,$$
$$\frac{2}{3}-\frac{4}{5}=\frac{2\cdot 5}{3\cdot 5}-\frac{4\cdot 3}{5\cdot 3}=\frac{10}{15}-\frac{12}{15}=\frac{-2}{15}\,.$$

It is, by the way, good practice to simplify the result as much as possible.

2.3 Multiplication and Division

The area of a rectangle is the product of its length and width. Therefore the product of two numbers may be interpreted as the area of the corresponding rectangle. From this we see that $\frac{4}{5}\cdot\frac{2}{3}$ must be the area of a rectangle, whose length is $\frac{4}{5}$ and whose width is $\frac{2}{3}$. Figure 2.4 shows the result of this multiplication; we have

$$\frac{4}{5}\cdot\frac{2}{3}=\frac{8}{15}\,.$$

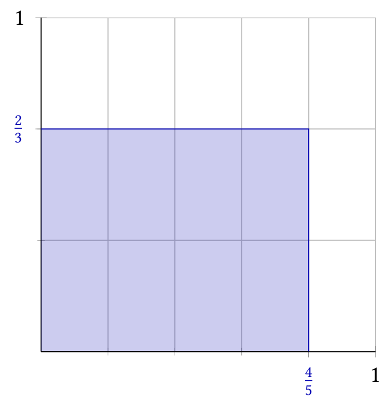


Figure 2.4: Here we see that $\frac{4}{5}\cdot\frac{2}{3}=\frac{8}{15}$.

Analysing the figure, we see that the number of coloured rectangles (the numerator of the result) is the product of the two numerators (4 and 2). The total number of small rectangles, which make up the area 1, can be found by multiplying the two denominators (5 and 3). This means that we have

$$\frac{4}{5}\cdot\frac{2}{3}=\frac{4\cdot 2}{5\cdot 3}=\frac{8}{15}\,.$$

From this argument we see that we multiply fractions by multiplying their numerators and their denominators.

Example 2.5 A few multiplications involving fractions:

$$\frac{3}{2}\cdot\frac{5}{7}=\frac{3\cdot 5}{2\cdot 7}=\frac{15}{14}\,,$$

$$\frac{4}{9} \cdot \frac{7}{11} = \frac{4 \cdot 7}{9 \cdot 11} = \frac{28}{99} ,$$
$$\frac{1}{3} \cdot \frac{5}{2} \cdot \frac{13}{7} = \frac{1 \cdot 5 \cdot 13}{3 \cdot 2 \cdot 7} = \frac{65}{42} .$$

The final arithmetical operation is division. We use the following calculation as an example:

$$\frac{4}{7} \Big/ \frac{2}{5} .$$

Before we analyse this calculation, we first note that [6]

$$\frac{2}{5} \cdot \frac{5}{2} = \frac{2 \cdot 5}{5 \cdot 2} = \frac{10}{10} = 1 .$$

[6]The fraction we get by exchanging the numerator and the denominator is called the *reciprocal* of the original fraction.

Now, if we multiply the original calculation $\frac{4}{7}/\frac{2}{5}$ by 1, we do not change the result. Therefore we write

$$\frac{4}{7} \Big/ \frac{2}{5} \cdot 1 .$$

But since $\frac{2}{5} \cdot \frac{5}{2} = 1$, we might as well write

$$\frac{4}{7} \Big/ \frac{2}{5} \cdot \left(\frac{2}{5} \cdot \frac{5}{2} \right) .$$

The order in which we multiply and divide does not matter. We can therefore move the parenthesis and write

$$\left(\frac{4}{7} \Big/ \frac{2}{5} \cdot \frac{2}{5} \right) \cdot \frac{5}{2} .$$

Inside the parenthesis, we now have $\frac{4}{7}$ divided by $\frac{2}{5}$ and then multiplied by $\frac{2}{5}$. Since division and multiplication are opposite operations, this does not change the number $\frac{4}{7}$. Therefore we might as well remove it and just write

$$\left(\frac{4}{7} \right) \cdot \frac{5}{2} ,$$

which is the same $\frac{4}{7} \cdot \frac{5}{2}$.

All the way through this argument, we have looked at the same calculation. We therefore conclude that

$$\frac{4}{7} \Big/ \frac{2}{5} = \frac{4}{7} \cdot \frac{5}{2} .$$

Thus we may change a division by a fraction into a multiplication by the reciprocal fraction.

Example 2.6 A few examples:

$$\frac{3}{8} \Big/ \frac{7}{5} = \frac{3}{8} \cdot \frac{5}{7} = \frac{15}{56} ,$$
$$\frac{1}{2} \Big/ \frac{11}{5} = \frac{1}{2} \cdot \frac{5}{11} = \frac{5}{22} ,$$
$$\frac{7}{6} \Big/ \frac{3}{13} = \frac{7}{6} \cdot \frac{13}{3} = \frac{91}{18} .$$

2.4 Fractions and Whole Numbers

If a calculation involves whole numbers as well as fractions, the easiest way to proceed is to write the whole numbers as fractions (with denominator 1), e.g.

$$8 = \frac{8}{1}, \qquad 12 = \frac{12}{1} \qquad \text{and} \qquad -3 = \frac{-3}{1}.$$

Now the calculations only involve fractions, and we may use the methods described above.

Example 2.7 In this example, we look at the calculation $2 + \frac{3}{4}$. If we write the number 2 as a fraction, we get

$$\frac{2}{1} + \frac{3}{4}.$$

Now we need a common denominator, so we multiply the numerator and the denominator of the first fraction by 4:

$$\frac{2}{1} + \frac{3}{4} = \frac{2 \cdot 4}{1 \cdot 4} + \frac{3}{4} = \frac{8}{4} + \frac{3}{4} = \frac{11}{4}.$$

The result of this addition is $\frac{11}{4}$.

Example 2.8 The result of the division $\frac{4}{3}\big/5$ can be found by writing the number 5 as the fraction $\frac{5}{1}$. Then we have

$$\frac{4}{3}\bigg/\frac{5}{1} = \frac{4}{3} \cdot \frac{1}{5} = \frac{4}{15}.$$

2.5 Sign

Calculations involving fractions with negative numbers in the numerator and/or the denominator are done in exactly the same way as divisions since a fraction is, in essence, a form of division.

If we divide a positive number by a negative number, or vice versa, the result is negative. Therefore we have

$$\frac{-6}{11} = \frac{6}{-11}.$$

Usually we write the sign outside the fraction, i.e.

$$-\frac{6}{11}.$$

A negative sign outside the fraction means that the fraction itself is negative. Since we get the same result if the negative sign is written on the numerator or the denominator, it does not matter where we write it (as long as there is *only one* negative sign).

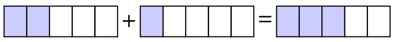
If both the numerator and the denominator have a negative sign, the resulting fraction is actually positive, i.e.

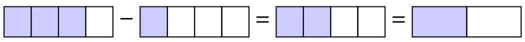
$$\frac{-13}{-7} = \frac{13}{7}.$$

If we are presented with a calculation involving numerous multiplications and divisions, we can determine the sign of the result by remembering that every pair of negative signs "vanishes". If the amount of negative signs is even, the result is therefore positive; if the amount of negative signs is odd, the result is negative.

2.6 Exercises

Exercise 2.1
Write down the calculations which the pictures represent:

a) 

b) 

Exercise 2.2
Calculate the following, and simplify as much as possible:

a) $\frac{3}{4} - \frac{1}{2}$ b) $\frac{5}{3} + \frac{7}{9}$

c) $\frac{3}{4} + \frac{1}{9}$ d) $\frac{3}{7} + \frac{11}{3}$

e) $\frac{2}{9} + \frac{5}{6}$ f) $\frac{15}{4} - \frac{1}{6}$

Exercise 2.3
Simplify these fractions as much as possible:

a) $\frac{24}{32}$ b) $\frac{112}{200}$

c) $\frac{63}{77}$ d) $\frac{17}{136}$

Exercise 2.4
Calculate the following, and simplify as much as possible:

a) $\frac{5}{8} - \frac{13}{20} - \frac{3}{15}$ b) $\frac{11}{9} - \frac{7}{18} + \frac{4}{27}$

c) $\frac{4}{5} + \frac{7}{3} - \frac{5}{4}$ d) $\frac{3}{6} - \frac{12}{16} - \frac{3}{8}$

e) $\frac{4}{9} - \frac{7}{6} + \frac{12}{9}$ f) $\frac{5}{7} + \frac{3}{21} - \frac{2}{14}$

Exercise 2.5
Calculate the following, and simplify as much as possible:

a) $\frac{2}{3} \cdot 5 + 7 \cdot \frac{1}{2}$ b) $-\frac{3}{2} \cdot 8 + \frac{\frac{7}{2}}{3}$

Exercise 2.6
Calculate the following, and simplify as much as possible:

a) $\frac{9}{14} + \frac{3}{4} - \frac{2}{7}$ b) $3 \cdot \frac{3}{13} - \frac{\frac{4}{7}}{2} + \frac{2}{3}$

c) $\frac{\frac{5}{3}}{4} + \frac{1}{8} \cdot \frac{5}{6} - 32 \cdot \frac{1}{7}$ d) $\frac{3}{4} - \frac{\frac{1}{5}}{3} \cdot \frac{2}{5} + \frac{4}{3}$

Powers and Roots 3

In this chapter, we look at some of the rules involving powers and roots. First, we look at powers where the exponent is a natural number. This is then expanded to exponents, which are negative numbers or fractions.

3.1 Integer Exponents

Raising a number to a power is defined in the following way:[1]

$$3^5 = \overbrace{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}^{5\text{ times}} .$$

Roots are the opposite operation, e.g.

$$\sqrt[4]{16} = 2, \qquad \text{because } 2^4 = 16 .$$

If we multiply two powers that have the same base, we may do this

$$7^2 \cdot 7^4 = \overbrace{\underbrace{7 \cdot 7}_{2\text{ times}} \cdot \underbrace{7 \cdot 7 \cdot 7 \cdot 7}_{4\text{ times}}}^{2+4=6\text{ times}} = 7^6 .$$

Dividing two powers that have the same base yields this:

$$\frac{4^5}{4^3} = \frac{4 \cdot 4 \cdot \cancel{4} \cdot \cancel{4} \cdot \cancel{4}}{\cancel{4} \cdot \cancel{4} \cdot \cancel{4}} = 4 \cdot 4 = 4^2 ,$$

i.e.

$$\frac{4^5}{4^3} = 4^{5-3} .$$

If we have two consecutive powers we get

$$(2^4)^3 = \overbrace{\underbrace{(2 \cdot 2 \cdot 2 \cdot 2)}_{4\text{ times}} \cdot \underbrace{(2 \cdot 2 \cdot 2 \cdot 2)}_{4\text{ times}} \cdot \underbrace{(2 \cdot 2 \cdot 2 \cdot 2)}_{4\text{ times}}}^{4 \cdot 3\text{ times}} = 2^{4 \cdot 3} = 2^{12} .$$

Now we have seen what happens when we combine powers with the same base. Next, we examine what happens when we combine powers with the same exponent.

Multiplying two powers with the same exponent yields e.g.

$$5^3 \cdot 2^3 = 5 \cdot 5 \cdot 5 \cdot 2 \cdot 2 \cdot 2 = 5 \cdot 2 \cdot 5 \cdot 2 \cdot 5 \cdot 2 = (5 \cdot 2) \cdot (5 \cdot 2) \cdot (5 \cdot 2) = (5 \cdot 2)^3 .$$

[1] In the calculation 3^5, we call the number 3 the *base* and the number 5 the *exponent*.

Division leads to

$$\frac{7^4}{3^4} = \frac{7 \cdot 7 \cdot 7 \cdot 7}{3 \cdot 3 \cdot 3 \cdot 3} = \frac{7}{3} \cdot \frac{7}{3} \cdot \frac{7}{3} \cdot \frac{7}{3} = \left(\frac{7}{3}\right)^4 .$$

All of these calculations may be generalised. Thus we obtain these 5 power rules:

Theorem 3.1

If m and n are two natural numbers, and a and b are two arbitrary numbers, we have

1. $a^m \cdot a^n = a^{m+n}$.
2. If a is not 0, and $m > n$, then $\frac{a^m}{a^n} = a^{m-n}$.
3. $(a^m)^n = a^{m \cdot n}$.
4. $a^n \cdot b^n = (a \cdot b)^n$.
5. If b is not 0, then $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$.

3.2 Rational Exponents

In this section, we look at exponents that are not positive integers. This raises an interesting question: We know what 5^4 means, but how do we interpret e.g. 2^{-7} or $3^{\frac{1}{4}}$?

We assign a meaning to calculations such as these by demanding that the rules in theorem 3.1 must be true, no matter which values we assign to the exponents. It turns out that it is only possible for the rules to be true, if we define powers with negative or fractional exponents in a certain way.

If, e.g., we calculate 5^0, rule 2 in theorem 3.1 yields

$$5^0 = 5^{2-2} = \frac{5^2}{5^2} = 1 .$$

The base here is 5, but it could have been any number. From a similar calculation we can just as easily show that $7^0 = 1$. We can therefore generalise this to any number.[2]

[2] Except 0—because we cannot divide by 0.

If the exponent is negative, we can use the same rule to obtain[3]

[3] Here we use that we have just shown that $5^0 = 1$, $6^0 = 1$, $43^0 = 1$, etc.

$$6^{-3} = 6^{0-3} = \frac{6^0}{6^3} = \frac{1}{6^3} .$$

This calculation can also be performed using other numbers. Thus we find e.g. $13^{-7} = \frac{1}{13^7}$. The argument works for any base but 0.

If the exponent is a fraction, we use rule 3 in theorem 3.1 to calculate e.g. $(8^{\frac{1}{3}})^3$. This yields

$$(8^{\frac{1}{3}})^3 = 8^{\frac{1}{3} \cdot 3} = 8^1 = 8 .$$

But we also know that[4]

[4] Because roots are the opposites of powers.

$$(\sqrt[3]{8})^3 = 8 .$$

Therefore

$$8^{\frac{1}{3}} = \sqrt[3]{8} .$$

This explains how to interpret the calculation if the exponent is $\frac{1}{2}$, $\frac{1}{7}$ or $\frac{1}{73}$; but it does not tell us what to do if the numerator of the fraction is not 1.

Here we instead look at the calculation

$$4^{\frac{5}{7}} = 4^{5\cdot\frac{1}{7}} = (4^5)^{\frac{1}{7}} = \sqrt[7]{4^5} .$$

If the power rules must apply to all numbers, we therefore need the following definition:

Definition 3.2

1. $a^0 = 1$ (if $a \neq 0$).
2. $a^{-n} = \frac{1}{a^n}$ (if $a \neq 0$).
3. $a^{\frac{p}{q}} = \sqrt[q]{a^p}$.

For rational exponents, we have the same rules as for integer exponents,[5] so we have the following theorem:

[5] It was this that led us to the definition 3.2.

Theorem 3.3

We have the following rules

1. $a^x \cdot a^y = a^{x+y}$.
2. $(a^x)^y = a^{x\cdot y}$.
3. $\frac{a^x}{a^y} = a^{x-y}$.
4. $a^x \cdot b^x = (a \cdot b)^x$.
5. $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$.

Since roots can be calculated by raising to a fractional power, we also have the following theorem:[6]

[6] The two rules follow from e.g.

$$\sqrt[3]{5} \cdot \sqrt[3]{2} = 5^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} = (5 \cdot 2)^{\frac{1}{3}} = \sqrt[3]{5 \cdot 2}$$
$$\frac{\sqrt[7]{4}}{\sqrt[7]{11}} = \frac{4^{\frac{1}{7}}}{11^{\frac{1}{7}}} = \left(\frac{4}{11}\right)^{\frac{1}{7}} = \sqrt[7]{\frac{4}{11}} .$$

Theorem 3.4

If $a > 0$ and $b > 0$ then

1. $\sqrt[x]{a} \cdot \sqrt[x]{b} = \sqrt[x]{a \cdot b}$.
2. $\frac{\sqrt[x]{a}}{\sqrt[x]{b}} = \sqrt[x]{\frac{a}{b}}$.

In theorem 3.4, there are no rules for the expressions $\sqrt[x]{a} \cdot \sqrt[y]{a}$ og $\frac{\sqrt[x]{a}}{\sqrt[y]{a}}$. This is because in these two cases it is always easier to write the roots as powers before reducing.

If we really want to, it is possible two deduce two formulas. This is left as an exercise for the reader.

3.3 Exercises

Exercise 3.1
Reduce the following expressions using power rules:

a) $5^2 \cdot 5^3$ b) $2^4 \cdot 2^5$

c) $\left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^2$ d) $\frac{3^6}{3^4}$

e) $\frac{(\frac{2}{3})^7}{(\frac{2}{3})^6}$ f) $\left(\frac{2}{3}\right)^4 \left(\frac{3}{2}\right)^4$

g) $2^6 \cdot 5^6$ h) $\frac{12^3}{2^3}$

i) $\frac{35^4}{5^4}$ j) $\frac{4^2}{(\frac{1}{2})^2}$

k) $(3^4)^5$ l) $\left(\left(\frac{2}{7}\right)^3\right)^3$

Exercise 3.2
In each of the following cases, determine whether the expression is defined (i.e. makes sense). Reduce the expression if this is the case.

a) $\sqrt[4]{16}$ b) $\sqrt[3]{-64}$

c) $\sqrt{(-4)^2}$ d) $\sqrt{-4^2}$

e) $\sqrt[3]{125}$ f) $\sqrt[3]{\frac{1}{8}}$

g) $\sqrt[4]{-81}$ h) $\sqrt[5]{\frac{1}{32}}$

i) $\frac{1}{\sqrt[3]{216}}$ j) $\sqrt[4]{\frac{1}{10000}}$

k) $\sqrt{8}\sqrt{32}$ l) $\sqrt{45}\sqrt{5}$

m) $\sqrt{-3}\sqrt{-3}$ n) $\frac{\sqrt{135}}{\sqrt{15}}$

o) $\sqrt{\frac{-8}{-2}}$ p) $\sqrt{-0}$

Exercise 3.3
Calculate the following roots if they exist:

a) $\sqrt{9} + \sqrt{16}$ b) $\sqrt{3^2 + 4^2}$

c) $\sqrt{3 \cdot 12}$ d) $\sqrt{13 + 3 \cdot 4}$

e) $\sqrt{-4^2 + 2^3}$ f) $\sqrt{12^2 + 5^2}$

g) $\sqrt[3]{(-2)^6}$ h) $\sqrt{(\frac{3}{5})^2 + (\frac{4}{5})^2}$

Exercise 3.4
Calculate the following numbers:

a) $49^{\frac{1}{2}}$ b) $\left(\frac{4}{9}\right)^{\frac{1}{2}}$

c) $27^{\frac{-1}{3}}$ d) 64^0

e) $-64^{\frac{1}{3}}$ f) $(3^2)^0$

g) $(-27)^{\frac{-1}{3}}$ h) $81^{\frac{1}{4}}$

i) $\left(\frac{1}{3}\right)^0$ j) 4^{-2}

k) $9^{-\frac{1}{2}}$ l) 100023^0

Exercise 3.5
Calculate the numbers

a) $-1 + (-1)^2 + (-1)^3 + \cdots + (-1)^{2006}$

b) $(-1)^2 + (-1)^4 + (-1)^6 + \cdots + (-1)^{2006}$

c) $(1 - 1 + 1)^6 - (-1 + 1 - 1)^4 - (1 - 1 + 1)^2$

Algebra 4

Algebra as a subject is about the rules that apply when we calculate. In mathematics we sometimes need to use numbers, whose values we do not know. These numbers are referred to as *unknowns*. Instead of the unknown number, we write a letter, e.g. x, y, a or A.[1]

[1] It is important to remember that we distinguish between upper and lower case letters—i.e. a and A are not the same number.

If a calculation involves unknowns, we cannot calculate a final result. But we may sometimes be able to reduce or simplify the calculation. This makes the calculation easier when we finally get to know the values of the unknowns.

Since e.g.

$$5 + 5 + 5 = 3 \cdot 5 \;,$$

we know that

$$x + x + x = 3 \cdot x \;,$$

no matter which value x has. Similarly, we have

$$8 \cdot 8 \cdot 8 \cdot 8 = 8^4 \;,$$

so

$$x \cdot x \cdot x \cdot x = x^4 \;.$$

Hence algebraic rules may be used to simplify calculations and formulas to make them easier to work with.

The algebraic rules we use here are no different from the usual arithmetical rules—i.e. for calculations involving known numbers. This is because the letters above *are* numbers. There is, however, one small difference: In algebraic expressions we do not necessarily write the multiplication sign if omitting it does not lead to confusion.[2] Therefore

[2] When calculating $7 \cdot 3$, the multiplication sign is necessary—since 73 is not the same as $7 \cdot 3$. But we do not need the sign when we write $7x$.

$$\begin{aligned}
4p &= 4 \cdot p \\
3xy &= 3 \cdot x \cdot y \\
5w^2 &= 5 \cdot w^2 \\
2y^3z &= 2 \cdot y^3 \cdot z \\
7ab^2 &= 7 \cdot a \cdot b^2 \\
2(x+y) &= 2 \cdot (x+y) \\
(5-x)(2-x) &= (5-x) \cdot (2-x) \;.
\end{aligned}$$

4.1 Like terms

If we add x and x, we get $2x$. Therefore the following is also correct:

$$3x + 4x = \overbrace{x + x + x}^{\text{3 terms}} + \overbrace{x + x + x + x}^{\text{4 terms}} = 7x \ .$$

If the letters are the same, we may add the numbers (or subtract them).

Example 4.1 A few examples of adding or subtracting like terms:

$$\begin{aligned} 2x + 5x &= 7x \\ 5p^2 + 11p^2 &= 16p^2 \\ 4y + 7y + 2y &= 13y \\ 8xy - 3xy &= 5xy \\ 7w^3 - 15w^3 &= -8w^3 \ . \end{aligned}$$

Terms such as $2x$ and $5x$ are called *like terms*. If we add two like terms, we add the numbers.[3] The terms are only like terms if the letters are *exactly* the same. This means we cannot add e.g. $2a$ and $4b$.

[3] If we subtract, we use a similar rule.

Example 4.2 Since we can only add like terms, we have

$$\begin{aligned} 3x - 8y + 6x &= 3x + 6x - 8y = 9x - 8y \\ 4w + 7u - w + 5uw &= 4w - w + 7u + 5uw = 3w + 7u + 5uw \\ -3y + 4z + 5y - z &= -3y + 5y + 4z - z = 2y + 3z \\ 4x + 3x^2 + 2x &= 4x + 2x + 3x^2 = 6x + 3x^2 \ . \end{aligned}$$

From this example, we see that $4x$ and $3x^2$ are not like terms. This is because the letters have to be *exactly* the same—and x and x^2 are not raised to the same power.[4]

[4] However, $3ab$ and ba are like terms, since the order of multiplication does not matter, i.e. $ba = ab$. The same argument also applies to e.g. xy^2 and y^2x; but not yx^2, since here the wrong number (x instead of y) is squared.

4.2 Parentheses

When we simplify expressions, we often use the following 3 rules, which apply to addition and multiplication.

The commutative law: $a + b = b + a$ og $a \cdot b = b \cdot a$.

The associative law: $a + (b + c) = (a + b) + c$ og $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

The distributive law: $a \cdot (b + c) = a \cdot b + a \cdot c$

The commutative law merely states that the order of addition or multiplication does not matter. The associative law states that some parentheses are irrelevant, e.g.

$$8x + (3x + 6x) = (8x + 3x) + 6x \ .$$

The parenthesis here is irrelevant. We might as well just write

$$8x + 3x + 6x \ .$$

The sum of these three terms is $17x$.

If a parenthesis is preceded by a "+", we can just remove the parenthesis. This is not true if it is preceded by a "−". Here we need the distributive law to find out how to proceed.

The distributive law follows from the argument sketched in figure 4.1, and it tells us how to multiply a number with a sum.

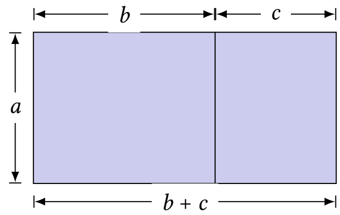


Figure 4.1: The area of the entire rectangle is $a \cdot (b + c)$, but we can also find the area as the sum of the areas of the two smaller rectangles, i.e. $a \cdot b + a \cdot c$. Since it is the same area, we must have $a \cdot (b + c) = a \cdot b + a \cdot c$.

If we remember that $-x = (-1)x$, we may deduce that

$$a - (b + c) = a + (-1)(b + c) = a + (-1)b + (-1)c = a - b - c \ .$$

If a parenthesis is preceded by a "−", we can therefore remove the parenthesis if we change the sign of every term inside the parenthesis.

Example 4.3 A few examples of how to remove parentheses:

$$\begin{aligned} x + (8 - 2x) &= x + 8 - 2x \ , \\ 8y - (y + 3) &= 8y - y - 3 \ , \\ 5t + (6 + 2t) &= 5t + 6 + 2t \ , \\ 7p - (1 - 6p) &= 7p - 1 + 6p \ . \end{aligned}$$

If we need to multiply a number and a sum, we also use the distributive law.

Example 4.4 Some examples of multiplying a number with a sum or a difference:

$$\begin{aligned} 2(x + 5) &= 2x + 2 \cdot 5 = 2x + 10 \ , \\ x - 8(5 + x) &= x + (-8) \cdot 5 + (-8)x = x - 40 - 8x \ , \\ y(3 + y) &= 3y + y^2 \ . \end{aligned}$$

A more advanced example might be:

$$\begin{aligned} 5 - ab(3b + a) &= 5 + (-ab) \cdot 3b + (-ab)a = 5 - ab \cdot 3b - aba \\ &= 5 - 3abb - aab = 5 - 3ab^2 - a^2b \ . \end{aligned}$$

It would also be of interest to know how to multiply two sums—i.e. a calculation such as $(a + b)(c + d)$. Here we use the distributive law twice to obtain

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd \ .$$

As we can see, every term in the first parenthesis is multiplied by every term in the last parenthesis. We can illustrate this in the following way:

$$(a + b)(c + d) = ac + ad + bc + bd \ .$$

All the rules involving parentheses are summed up in the following theorem:

Theorem 4.5

For calculations involving parentheses we have:

1. $a + (b + c) = a + b + c$.
2. $a - (b + c) = a - b - c$.
3. $a(b + c) = ab + ac$.
4. $(a + b)(c + d) = ac + ad + bc + bd$.

Factoring

Sometimes it is a good idea to use the distributive law "backwards". Theorem 4.5(3) then becomes

Theorem 4.6

For the three numbers a, b and c, we have

$$ab + ac = a(b + c) \ .$$

Rewriting an expression in this way is called *factoring*. We begin by identifying a term, which divides every term in the expression. E.g.

$$12x + 18y = 6 \cdot 2x + 6 \cdot 3y = 6(2x + 3y) \ .$$

Here 6 divides every term in the original expression.

Example 4.7 Examples of factoring could be

$$\begin{aligned} 5x + 15z &= 5x + 5 \cdot 3y = 5(x + 3y) \ , \\ 7a + ab &= a(7 + b) \ , \\ 3pq - 5pq^2 &= 3pq - 5pq \cdot q = pq(3 - 5q) \ . \end{aligned}$$

The advanced example where we factor out $2xy$, is

$$\begin{aligned} 2x^2y + 4xy^2 - 6xy &= 2xxy + 2 \cdot 2xyy - 3 \cdot 2xy \\ &= 2xy \cdot x + 2xy \cdot 2y - 2xy \cdot 3 = 2xy(x + 2y - 3) \ . \end{aligned}$$

Factoring is a useful tool in many situations. As an example we look at a fraction that can be simplified after factoring:

$$\frac{6x + 9}{12} = \frac{3(2x + 3)}{12} = \frac{3(2x + 3)/3}{12/3} = \frac{2x + 3}{4} \ .$$

4.3 Quadratic Multiplication Formulas

When we multiply two parentheses, we multiply every term in the first parenthesis with every term in the last. If some of the terms are equal, we can simplify the resulting expression.

Two examples are[5]

[5]In theses calculations we use throughout that $ab = ba$.

$$(a+b)^2 = (a+b)(a+b) = aa + ab + ba + bb = a^2 + b^2 + 2ab\ ,$$

and

$$(a-b)^2 = (a-b)(a-b) = aa + a(-b) + (-b)a + (-b)(-b) = a^2 + b^2 - 2ab\ .$$

If we add the terms in the first parenthesis and subtract them in the last, we get

$$(a+b)(a-b) = aa + (-b)a + ba + (-b)b = a^2 - b^2\ .$$

Collectively, these calculations yield the following theorem:

Theorem 4.8

A square of a sum:

1. $a^2 + b^2 + 2ab = (a+b)^2$.

A square of a difference:

2. $a^2 + b^2 - 2ab = (a-b)^2$.

A difference of two squares:

3. $a^2 - b^2 = (a+b)(a-b)$.

Example 4.9 The formulas may be used in this manner:

$$x^2 + 49 + 14x = x^2 + 7^2 + 2 \cdot 7x = (x+7)^2\ ,$$
$$4p^2 - 25q^2 = (2p)^2 - (5q)^2 = (2p+5q)(2p-5q)\ ,$$
$$9a^2 + 36 - 36a = (3a)^2 + 6^2 - 2 \cdot 3a \cdot 6 = (3a-6)^2\ .$$

Example 4.10 Sometimes we can simplify fractions, even if at first it looks impossible:

$$\frac{x^2 + 25 - 10x}{4x - 20} = \frac{(x-5)^2}{4(x-5)} = \frac{x-5}{4}\ .$$

4.4 Exercises

Exercise 4.1
Find the terms and the factors in each of the 3 expressions:

a) $2x + 4y - xy$

b) $7p^2 + 16pqz - 25$

c) $17xyz + x^{113}y - 17 + y^4 + xy^2$

Exercise 4.2
Simplify the following expressions as much as possible:

a) $4(a+5b) + 7(3a+b)$

b) $x(y+3) - y(x-6)$

c) $17x - (14x - (7x+2))$

d) $(6x-5y)(3x-10) - (2x+7)(9x+4y)$

e) $4(2x+7y) - 5(-2y+4x) + 3(-3x-5y)$

4.5 Exercises

Exercise 4.3
Simplify the following expressions as much as possible:

a) $2(x+3)-4x$
b) $6(7+a)-(3-a)$
c) $12(a-b)+6(5b-a)$
d) $27(c-3z)-(-4+c)$
e) $-2(c+h^2)+h(4-c)$
f) $\frac{12x+6y}{3}-7y$
g) $\frac{6x+xy}{x}+2(y+3)$

Exercise 4.4
The 6 expressions below have been rewritten. In each case, the same rule has been applied. Describe what is happening, and write down a general rule based on your observations.

a) $3x+3y=3(x+y)$
b) $6a+3b=3(2a+b)$
c) $7c+14=7(c+2)$
d) $14y+7=7(2y+1)$
e) $2x+4y^2+18z=2(x+2y^2+9z)$
f) $3ab+6ab^2+9a^2b=3ab(1+2b+3a)$

Exercise 4.5
Factor the expressions as much as possible:

a) $3x-x^2$
b) $12p^2-6pq$
c) $a^2b-3a+7ab^2$
d) $4xy-6x^2y+2xy^2$

Exercise 4.6
Simplify the following expressions as much as possible:

a) $4(2x+7y)-5(-2y+4x)+3(-3x-5y)$
b) $6(4a+7b-5)-3(8a+8b-2)+(a-2b+3)\cdot 9$
c) $\frac{a}{2}+\frac{5a}{8}+\frac{3a}{4}$.
d) $\frac{18x+2}{4y}-\frac{5x-1}{3y}$.
e) $\frac{14a^2b^3c}{21a^5bc^4}$

Exercise 4.7
Simplify the following expressions as much as possible:

a) $4x-6y+2(x+3y)$
b) $x(x+2y)-(x+y)^2$
c) $\frac{12x^2y^4}{3xy}$

Exercise 4.8
Simplify the following expressions as much as possible:

a) $(2x-3)\cdot 5+9\cdot(y-x)+3\cdot(5-3y)$
b) $\frac{25x^2-5x}{10x^2}$
c) $4b\cdot\left(\frac{a}{b}\right)^2+\frac{5a}{b}$

Exercise 4.9
Simplify the following fractions as much as possible:

a) $\frac{10a^2bc^3}{4a^4b^2c}$
b) $\frac{7x^2y^4z}{49x^3yz^2}$

Exercise 4.10
Simplify the following expressions as much as possible:

a) $\frac{11x}{18}+\frac{5x}{6}+\frac{2x}{9}$
b) $\frac{3p}{3q}+\frac{p}{8q}-3$
c) $\frac{a-(2b-9)}{a-2b}-\frac{a-(b-5)}{a-2b}+\frac{a+(b-4)}{a-2b}$

Exercise 4.11
Calculate the following squares:

a) $(x+7)^2$
b) $(2x-3)^2$
c) $(3q+2)^2$
d) $(5p^2-3x)^2$

5 Equations

An equation consists of two calculations separated by an equals sign. The equals sign may be viewed as the statement that the two calculations yield the same result.

An unknown number (the *unknown*)[1] is present in at least one of the two calculations. A *solution* to the equation is a number, such that the statement is true when the number is inserted in place of the unknown.

[1]An equation might contain more than one unknown, but in the simplest case there is only one.

Example 5.1 An equation could be

$$5x - 9 = 2x \; .$$

The two calculations are $5x - 9$ and $2x$. The equation states that these calculations yield the same result.

$x = 3$ is a solution to the equation, since the two sides of the equation yield

$$5 \cdot 3 - 9 = 6 \quad \text{(left hand side, } 5x - 9\text{)},$$
$$2 \cdot 3 = 6 \quad \text{(right hand side, } 2x\text{)},$$

when we insert 3 in place of x. I.e. the two calculations yield the same result (6), when $x = 3$.

On the other hand, $x = 7$ is not a solution, since

$$5 \cdot 7 - 9 = 36 \; ,$$
$$2 \cdot 7 = 14 \; .$$

Here the two sides yield different results.

5.1 Solving an Equation

Solving an equation consists of finding those numbers that are solutions to the equation.[2] This involves a simple technique.

[2]It is possible for equations to have more than one solution; it is also possible to have equations that have no solutions.

The two sides of the equations are calcutions that yield the same result if we insert a solution in place of the unknown. E.g.

$$2 \cdot 4 + 3 = 11 \qquad \text{and} \qquad 5 \cdot 4 - 9 = 11 \; ,$$

which means that $x = 4$ is a solution to the equation[3]

$$2x + 3 = 5x - 9 \; . \tag{5.1}$$

[3]Both sides yield 11 when we insert 4 in place of x.

But the equation

$$2x + 3 + 9 = 5x - 9 + 9$$

must have the same solution. The calculations are not the same as before, so each side no longer yields 11; but the results on either side are still equal, since we added the same number to both sides. Now when we insert $x = 4$, we get

$$2 \cdot 4 + 3 + 9 = 20 \qquad \text{and} \qquad 5 \cdot 4 - 9 + 9 = 20 \; .$$

So if we add the same number to both sides of an equation, we get a new equation—but one with the same solution as the previous equation.

This reasoning also works for subtraction, multiplication, etc. We therefore have the following theorem:

Theorem 5.2

If we carry out the same arithmetical operation on both sides of an equation, we get a new equation with the exact same solutions.

The equation (5.1) might be solved in the following manner:

$2x + 3 = 5x - 9$	Equation (5.1)
$2x + 3 + 9 = 5x - 9 + 9$	Add 9 to both sides.
$2x + 12 = 5x$	Reduce
$2x + 12 - 2x = 5x - 2x$	Subtract $2x$ from both sides.
$12 = 3x$	Reduce.
$\frac{12}{3} = \frac{3x}{3}$	Divide by 3 on both sides.
$4 = x$	Reduce.

The last line is also an equation, but is an equation that is easy to solve. The solution to the equation $4 = x$ is $x = 4$—and this is also the solution to the original equation.[4]

We may use whichever operation we want to, but it is absolutely necessary that we always use the exact same operation on both sides of the equation.[5]

When we use an operation on both sides of an equation, it is important to remember to use the operation on the *entire* side and not just a part of it. See the examples below.

Example 5.3 If we want to multiply by 2 on both sides of the equation $\frac{1}{2}x + 3 = 8$, we need to use parentheses:

$$\begin{aligned} \tfrac{1}{2}x + 3 &= 8 && \iff^{6} \\ 2 \cdot \left(\tfrac{1}{2}x + 3\right) &= 2 \cdot 8 && \iff \\ x + 6 &= 16 \; . \end{aligned}$$

If we finish solving the equation, we find the solution $x = 10$.

[4]The point of adding, subtracting etc. is to get to an equation, which gives us the solutions directly.

[5]It is, however, never permitted to multiply by 0, since then the equation is reduced to $0 = 0$, which is always correct—and then it is not possible to find solutions to the original equation.

[6]The sign $\iff$ means "if and only if". This means that the two statements on each side of the arrow are logically equal, i.e. one of them is true only if the other one is true and vice versa.

Example 5.4 If we want to solve the equation $x^2 + 4 = 13$, we might be tempted first to take the square root on both sides. This yields

$$x^2 + 4 = 13 \qquad \Longleftrightarrow \qquad \sqrt{x^2 + 4} = \sqrt{13}\ .$$

Here, we cannot reduce the left hand side, since we need to add before we take the square root—and we cannot do this, because we do not know the value of x.

It turns out that it is a better idea to first subtract 4 to get

$$x^2 + 4 = 13 \qquad \Longleftrightarrow \qquad x^2 = 9\ .$$

This equation is easy to solve. Its two solutions are $x = -3$ and $x = 3$.[7]

[7]Remember that $(-3)^2 = 9$, since the product of two negative numbers is positive. Therefore $x = -3$ is also a solution to the equation.

Testing a Solution

If we are given a solution to an equation, or we want to check whether a solution is correct, we may test the solution. We simply insert the solution into both sides of the equation and see if we get the same result.

Example 5.5 Is $x = 2$ a solution to $x^3 - 3 = 2 \cdot x + 1$?

The left hand side yields

$$2^3 - 3 = 8 - 3 = 5\ .$$

The right hand side yields

$$2 \cdot 2 + 1 = 4 + 1 = 5\ .$$

When we insert $x = 2$, the two sides yield the same result. Therefore $x = 2$ *is* a solution to the equation.

Example 5.6 Is $x = 3$ a solution to the equation $\frac{4x}{x+1} = 5$?

The left hand side yields

$$\frac{4 \cdot 3}{3 + 1} = \frac{12}{4} = 3\ .$$

This is not equal to 5, which is the right hand side. Therefore $x = 3$ is *not* a solution.

How about $x = -5$? Here the left hand side yields

$$\frac{4 \cdot (-5)}{-5 + 1} = \frac{-20}{-4} = 5\ .$$

This is equal to the right hand side, so $x = 5$ is a solution.

5.2 The Zero Product Rule

If we multiply by 0, the result is always 0. On the other hand, we cannot multiply two non-zero numbers and get 0 as a result. Therefore, if the result of a multiplication is 0, at least one of the numbers involved must be 0. This leads us to the following theorem:

Theorem 5.7: The Zero Product Rule

If a product is 0, at least one of the factors is 0:

$$a \cdot b = 0 \quad \Longleftrightarrow \quad a = 0 \lor b = 0 \,.$$

If one side of an equation is 0, and the other side is a product, we can use this theorem to solve the equation.

Example 5.8 What are the solutions to the equation $(x - 3) \cdot (x + 2) = 0$?

On the right hand side, we have 0, and on the left hand side the product of $x - 3$ and $x + 2$. According to the zero product rule, at least one of these factors must be 0, i.e.

$$x - 3 = 0 \quad \text{or} \quad x + 2 = 0 \,,$$

which leads to the solutions

$$x = 3 \quad \text{or} \quad x = -2 \,.$$

Example 5.9 The equation $(x + 2)(x - 4)(x + 1) = 0$ can be solve using the zero product rule:[8]

$$\begin{aligned} (x + 2)(x - 4)(x + 1) = 0 \quad & \Longleftrightarrow \\ x + 2 = 0 \lor x - 4 = 0 \lor x + 1 = 0 \quad & \Longleftrightarrow \\ x = -2 \lor x = 4 \lor x = -1 \,. & \end{aligned}$$

[8] The sign $\lor$, which we use below, means "or".

Sometimes we can use the zero product rule if we are able to rewrite one side of the equation as a product.

Example 5.10 The equation $x^2 - 5x = 0$ can be solved in this manner:

First we factor out x

$$x \cdot (x - 5) = 0 \,,$$

and then we use the zero product rule

$$x = 0 \lor x - 5 = 0 \,.$$

Therefore the equation has the two solutions

$$x = 0 \lor x = 5 \,.$$

5.3 Two Equations in Two Unknowns

In the previous two sections, we only looked at equations in one unknown. An example of an equation in more unknowns is

$$3x - y = 4 \,.$$

Here there are two unknowns, x and y. If we have one equation in two unknowns, there is an infinite amount of pairs (x, y) of solutions to the equation.

E.g.

$$x = 5, y = 11 : \qquad 3 \cdot 5 - 11 = 4$$
$$x = 1, y = -1 : \qquad 3 \cdot 1 - (-1) = 4 .$$

But if we have *two* equations in two unknowns, there is exactly one pair of numbers, which solve both equations.[9]

[9]There are a few exceptions, which are described below.

Example 5.11 The two equations

$$5x - y = 3 \quad \text{and} \quad 2x + 4y = 10$$

have the solution $x = 1$ and $y = 2$ because

$$5 \cdot 1 - 2 = 3 \quad \text{and} \quad 2 \cdot 1 + 4 \cdot 2 = 10 .$$

No other values of x and y solve both equations.

Two equations in two unknowns is also called a *system of equations*. The system of equations in the example above has only one solution. Some systems of equations have more than one solution or no solution at all.

Example 5.12 The two equations

$$x + y = 2 \quad \text{and} \quad 3x + 3y = 6 ,$$

have an infinite amount of solutions.

This is the case because the second equation is actually the same as the first multiplied by 3. Therefore the two equations have the exact same solutions, and a pair (x, y), which solves the first equation, also solves the second.

Solving a system of equations consists of finding the pair(s) of numbers, which solve(s) the system. Below we describe two methods.

The Method of Substitution

Solving two equations in two unknowns via the method of substitution is done by solving for one of the unknowns in the first equation and inserting the found expression in the second. Doing this results in an equation with just one unknown.

Example 5.13 To solve the system of equations

$$2x + y = 7 \quad \text{og} \quad 5x - 3y = 12 ,$$

we solve for y in the first equation. We get

$$2x + y = 7 \quad \Longleftrightarrow \quad y = 7 - 2x . \qquad (5.2)$$

Next, we insert this expression into the second equation

$$5x - 3y = 12 \quad \Longrightarrow \quad 5x - 3\big(7 - 2x\big) = 12 .$$

We now solve this new equation:

$$\begin{aligned} 5x - 3(7 - 2x) &= 12 \\ 5x - 21 + 6x &= 12 \\ 11x - 21 &= 12 \\ 11x &= 12 + 21 \\ 11x &= 33 \\ x &= 3 \ . \end{aligned}$$

From (5.2), we have $y = 7 - 2x$, i.e.

$$y = 7 - 2 \cdot 3 = 1 \ .$$

Thus the solution to this system of equations is $x = 3$ and $y = 1$.

Example 5.14 The system[10]

$$x + y = 5 \quad \wedge \quad y^2 = 9 \ ,$$

can be solve by first solving the last equation:

$$y^2 = 9 \quad \Longleftrightarrow \quad y = -3 \vee y = 3 \ .$$

Each of these values of y have a corresponding value of x.

The first equation may be written as $x = 5 - y$, which gives us these two values of x:

$$\begin{aligned} y = -3 \quad &\Longrightarrow \quad x = 5 - (-3) = 8 \\ y = 3 \quad &\Longrightarrow \quad x = 5 - 3 = 2 \ . \end{aligned}$$

Therefore the system of equations has the following solution:

$$\big(x = 2 \ \wedge \ y = 3\big) \quad \vee \quad \big(x = 8 \ \wedge \ y = -3\big) \ .$$

[10]The sign $\wedge$ used below means "and". This is an "inclusive and", which means that the two equations on each side of $\wedge$ must be true simultaneously.

The Method of Elimination

Another method for solving two equations in two unknowns is the so-called "method of elimination". This method only works if both equations can be written in the form

$$ax + by = c \ ,$$

where a, b og c are three numbers.

The general idea is to rewrite the system of equations, such that either x or y has the same coefficient in the two equations. When we have done that, we can subtract the two equations to get a new equation with just one unknown (*eliminating* the other).

We illustrate this method through some examples

Example 5.15 Here we have the system of equations

$$\begin{cases} 3x + y = 11 \\ -2x + 5y = 21 \end{cases} .$$

We now multiply the first equation by 5 on both sides. Then we get

$$\begin{cases} 15x + 5y = 55 \\ -2x + 5y = 21 \end{cases} .$$

If we subtract these two equations[11] we get the new equation

$$(15x + 5y) - (-2x + 5y) = 55 - 21 ,$$

which we can reduce to get

$$17x = 34 .$$

The solution to this equation is $x = 2$.

Now we know the value of x, so we insert this into one of the equations from the original system. Here we choose $3x + y = 11$:

$$3 \cdot 2 + y = 11 \quad \Longleftrightarrow \quad y = 5 .$$

Thus the solution is $x = 2$ and $y = 5$.

In the above example, it was enough to rewrite one of the equations. Sometimes we need to rewrite them both.

Example 5.16 We rewrite the system of equations

$$\begin{cases} 5x - 4y = 22 \\ -2x + 8y = 4 \end{cases}$$

by multiplying the first equation by 2 and the second by 5:[12]

$$\begin{cases} 10x - 8y = 44 \\ -10x + 40y = 20 \end{cases} .$$

The coefficients of x differ in signs, so we add the equations instead of subtracting them:

$$(10x - 8y) + (-10x + 40y) = 44 + 20 .$$

We reduce this equation and solve it:

$$32y = 64 \quad \Longleftrightarrow \quad y = 2 .$$

We insert this value into one of the original equations, $5x - 4y = 22$:

$$5x - 4 \cdot 2 = 22 \quad \Longleftrightarrow \quad 5x = 30 \quad \Longleftrightarrow \quad x = 6 .$$

Therefore the solution to this system of equations is $x = 6$ and $y = 2$.

[11] We are allowed to subtract two equations because the left and right hand sides of an equation are representations of the same number. Thus we actually subtract the same number from both sides.

[12] We multiply each equation by the coefficient of x from the other equation.

5.4 Exercises

Exercise 5.1
Solve the equations

a) $7x - 3 = 9 + x.$
b) $4 \cdot (2x - 3) = 12.$
c) $4x - 3 = 8x - 19.$
d) $-x + 12 = \frac{x}{3}.$
e) $\frac{x-1}{4} = x + 5.$
f) $2(3x + 4) = 4x - 2.$

Exercise 5.2
Solve the equations

a) $(3x + 18) - 7 = (5x + 1) - 4$
b) $6x - (x + 5) = 3x + (x - 8)$
c) $3(t - 4) = 2t + 6$
d) $8(x - 8) = 2x + 8$
e) $3 + (2s - 5) = 6$
f) $3(q + 3) = 2(4 + q)$
g) $5 - (8x - 7) + 18x = 31x - (90 + 28x) - 45$

Exercise 5.3
Solve the equations

a) $\frac{20}{x} = 5$
b) $\frac{10}{x} + 3 = 8$
c) $\frac{8}{x} - 7 = 17$
d) $\frac{20}{x} = 5$
e) $\frac{9}{x-1} = 3$
f) $\frac{16}{x+5} - 3 = 1$

Exercise 5.4
Solve the equations

a) $\frac{x}{3} + 4 = 7$
b) $7 - \frac{y}{2} = 8$
c) $\frac{x}{2} + \frac{x}{3} = 10$
d) $\frac{x}{2} + \frac{x}{5} = 14$
e) $\frac{2y}{3} - \frac{y}{4} = 15$
f) $\frac{3y}{5} - \frac{y}{10} = 15$

Exercise 5.5
Isolate q in formula $a = 3q - qa + 5$. For which value of a is this impossible?

Exercise 5.6
Solve the equations

a) $x^3 = 27$
b) $x^2 = 64$
c) $x^5 = 1,61051$
d) $x^7 = -1$
e) $x^4 = 67$
f) $x^3 = -13$

Exercise 5.7
Solve the equations

a) $x^5 - 3 = 29$
b) $x^2 + 4 = 20$
c) $5x^3 = 320$
d) $0,1x^4 = 240,1$
e) $2x^3 - 5 = 11$
f) $6x^2 + 4 = 76$

Exercise 5.8
Solve the following equations:

a) $x(x + 2) = 0$
b) $(x + 3)(x - 1) = 0$
c) $2(x + 7)x = 0$
d) $(2x - 4)(3x + 12) = 0$
e) $(x + 6)(x - 1)(3x + 6)x = 0$
f) $x^2 - 6x = 0$

Exercise 5.9
Solve each of the following systems of equations:

a) $\begin{aligned} x - y &= 2 \\ x + y &= 10 \end{aligned}$
b) $\begin{aligned} x + 5y &= 5 \\ y &= 2 \end{aligned}$
c) $\begin{aligned} x + 3y &= 4 \\ 2x - 4y &= -2 \end{aligned}$
d) $\begin{aligned} 2x - 3y &= 4 \\ -3x + 2y &= -1 \end{aligned}$
e) $\begin{aligned} x + 3y &= 4 \\ 2x - 4y &= -2 \end{aligned}$
f) $\begin{aligned} 2a + 4b &= -2 \\ 7a - 3b &= 44 \end{aligned}$

Exercise 5.10
Which number yields the same result when added to 7 as when multiplied by 7?

Bibliography

[1] Florian Cajori. *A History of Mathematical Notation.* Vol. I: Notations in Elementary Mathematics. La Salle, Illinois: Open Court Publishing Company, 1974.

[2] Leo Rogers. *The History of Negative Numbers.* University of Cambridge. URL: http://nrich.maths.org/5961 (visited on 06/24/2014).